LECTURE 22: CONGRUENCE OF SURFACES (§6.9)

Defn/ M and $\bar{M}$ are <u>congruent</u> if $\exists F: \mathbb{R}^3 \to \mathbb{R}^3$ a Euclidean isometry for which $F(M) = \bar{M}$

<u>Example</u>: $z = x^2$ and $z = y^2 + 1$ (same shape, just rotate and shift-up 1)

<u>Example</u>: P_1 and P_2 planes in $\mathbb{R}^3$

<u>Example</u>: $M: z = xy$, $\bar{M}: z = \frac{x^2 - y^2}{2}$ (O'neill p. 314)

Congruent surfaces have same shape so the following is not surprising,

Thm/ If F is a Euclidean isometry with $F(M) = \bar{M}$ then $F_M = F/_M : M \to \bar{M}$ is an isometry of surfaces <u>and</u> $F_*(S(v)) = \bar{S}(F_*(v))$ for all tangent vectors v to M

<u>Proof</u>: see p. 315–316. Essentially this follows from fact restriction of smooth map is smooth here and push-forward of velocity is velocity of transformed curve... and shape S behaves nicely with F_*...

Thm/ Let M and $\bar{M}$ be oriented surfaces in $\mathbb{R}^3$. Let $F: M \to \bar{M}$ be an isometry that preserves orientation and shape operators $[F_*(S(v)) = \bar{S}(F_*(v))]$ then M and $\bar{M}$ are congruent. Moreover, $\exists \tilde{F}: \mathbb{R}^3 \to \mathbb{R}^3$ a Euclidean isometry such that $\tilde{F}/_M : M \to \bar{M}$ is just F; $\tilde{F} = F$.

<u>Proof</u>: See p. 317 – 318, this ultimately rests on Th^{m} 5.7 of Chapter 3 which we did not prove, but, nothing here is much different than the techniques & arguments we've covered. //

<u>Comment</u>: intuition behind theorem, or to connect to our curve discussion,

$\boxed{M \text{ isometric to } \bar{M}} \longleftrightarrow \boxed{\alpha \text{ and } \beta \text{ are both unit-speed defined on same interval}}$

$\boxed{M \text{ and } \bar{M} \text{ have same shape operators}} \longleftrightarrow \boxed{\kappa_\alpha = \kappa_\beta \ \& \ \tau_\alpha = \tau_\beta}$