LECTURE 2

Def^n/ $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R} = \{(p^1, p^2, \ldots, p^n) \mid p^i \in \mathbb{R}\}$

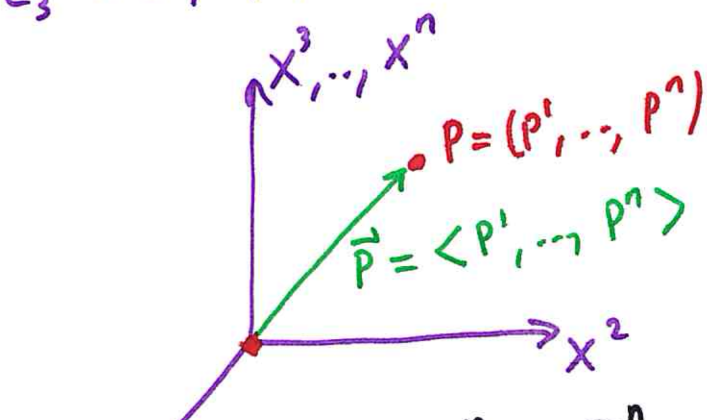
$$(P+Q)^i = P^i + Q^i$$
$$(cP)^i = cP^i$$

Def^n/ $(e_i)^j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

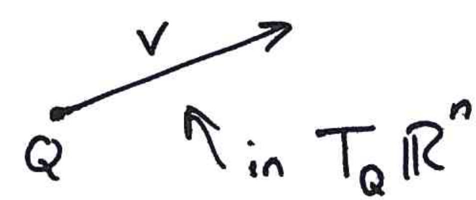
Ex/ $\mathbb{R}^3$: $e_1 = (1,0,0)$ $\quad P = (p^1, p^2, p^3)$

$e_2 = (0,1,0)$ $\quad = p^1(1,0,0) + p^2(0,1,0) + p^3(0,0,1)$

$e_3 = (0,0,1)$



$T_{(0,0,\ldots,0)}\mathbb{R}^n = \mathbb{R}^n$



$(Q, v) \in T_Q\mathbb{R}^n$

v_Q

$$T_p\mathbb{R}^n = \{(p,v) \mid v \in \mathbb{R}^n\} = \{p\} \times \mathbb{R}^n$$

$$T\,\mathbb{R}^n = \bigcup_{p\in\mathbb{R}^n} \{p\} \times T_p\mathbb{R}^n$$

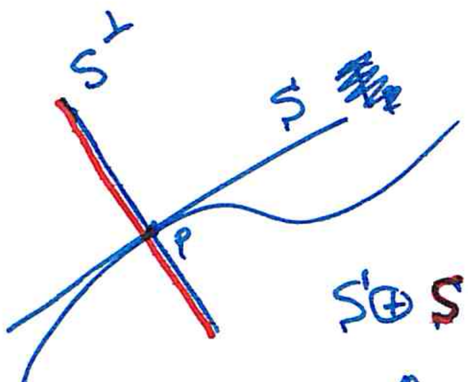
(p, v)

point of attachment $\nearrow$ $\quad$ $\nwarrow$ vector part

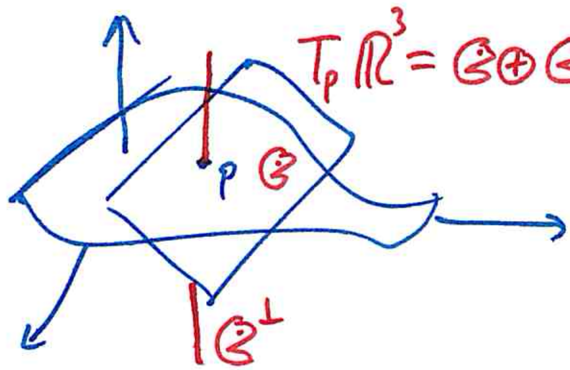
$$(p,v)+(p,w) = (p, v+w)$$
$$c(p,v) = (p, cv)$$
$$(p,v)\cdot(p,w) = v\cdot w = v^T w$$
$$\|(p,v)\| = \|v\| = \sqrt{v\cdot v}$$

(p,v), (p,w) are <u>orthogonal</u> if $v\cdot w = 0$

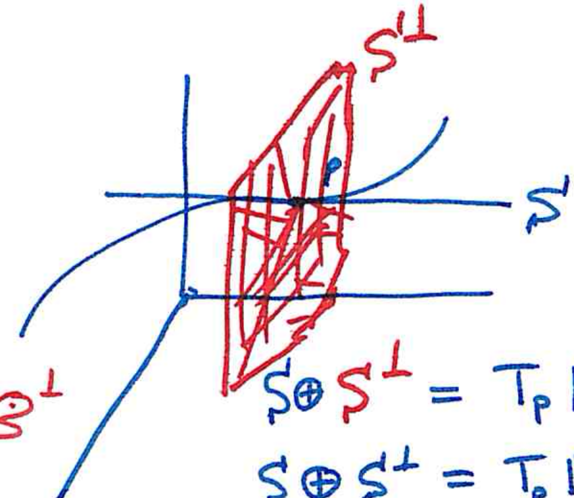
$S \subset T_p\mathbb{R}^n$, $\quad S^\perp = \{(p,v) \mid (p,v)\cdot(p,s) = 0 \ \forall (p,s) \in S\}$



$S \oplus S^\perp = T_p\mathbb{R}^2$



$T_p\mathbb{R}^3 = \mathcal{S} \oplus \mathcal{S}^\perp$

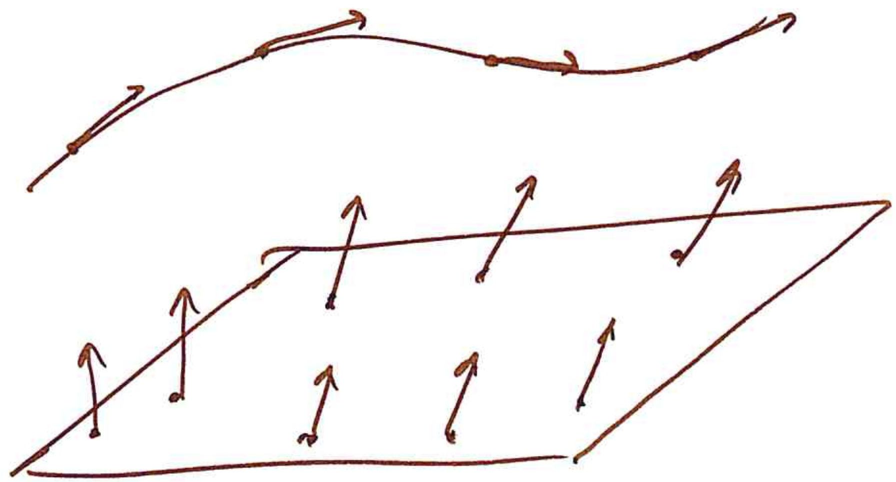


$S \oplus S^\perp = T_p\mathbb{R}^3$

$S \oplus S^\perp = T_p\mathbb{R}^n$

1-dim'l $\quad$ $(n-1)$-dim'l

VECTOR FIELD



$S \subseteq \mathbb{R}^n$

$X : S \longrightarrow T\mathbb{R}^n$

$X(p) \in \underbrace{\{p\} \times \mathbb{R}^n}_{\text{for each } p \in S} = T_p\mathbb{R}^n$

$\pi(p,v) = p \quad \forall (p,v) \in T\mathbb{R}^n$

$\pi(X(p)) = p \quad \forall p \in S$

$\underbrace{\pi \circ X = id_S}$

X a section of $TS \subset T\mathbb{R}^n$

§1.2 on tangent & cotangent spaces

Def^n/ $x^i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $x^i(p) = p^i$ for $i = 1, 2, \dots, n$

$\boxed{\mathbb{R}^3}$
$$\left.\begin{aligned} x(p) &= p^1 \\ y(p) &= p^2 \\ z(p) &= p^3 \end{aligned}\right\}$$ coordinate functions on $\mathbb{R}^3$

Ex/ $f = x^2 + yz$

$f(a,b,c) = a^2 + bc.$

Ex/ $f = \sum_{i=1}^{n} (x^i)^2$

$f(p) = \|p\|^2$

Def^n/ $f : \mathrm{dom}(f) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, smooth, $p \in \mathrm{dom}(f)$

The directional derivative of f at p w.r.t. $(p,v) \in T_p\mathbb{R}^n$

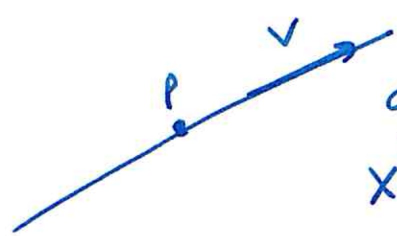
$$(Df)(v)(p) = (Df)(p,v) = \lim_{t \to 0}\left(\frac{f(p+tv) - f(p)}{t}\right) = \lim_{t \to 0}\left(\frac{f(\alpha(t)) - f(\alpha(0))}{t}\right)$$

$$= (f \circ \alpha)'(0)$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\underbrace{\alpha(0)}_{p})\underbrace{\frac{d x^i(\alpha(t))}{dt}}_{v^i}$$

$$= \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(p)$$

$$= v \cdot (\nabla f)(p)$$



$\alpha(t) = p + tv$

$x^i(\alpha(t)) = p^i + tv^i$

Prop. 1.2.5 $(Df)\underbrace{(p,v)}_{v_p} = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(p) = \Big(\underbrace{\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\Big|_p}_{v_p}\Big)(f)$

Def^n $T_p\mathbb{R}^n = \left\{ \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j}\Big|_p \;\middle|\; (v^1,\dots,v^n) \in \mathbb{R}^n \right\}$

$$(Df)(v_p) = v_p[f] \leftarrow v_p = \underbrace{v^1 \frac{\partial}{\partial x^1}\Big|_p + \dots + v^n \frac{\partial}{\partial x^n}\Big|_p}_{\text{a derivation}} = X$$

Prop 1.2.7 $X \in T_p\mathbb{R}^n$

$X[fg] = X[f]g(p) + f(p)X(g)$

$X[f+g] = X[f] + X[g]$

$X[cf] = c\,X[f]$

Proof: apply properties of partial derivatives to def^n of $T_p\mathbb{R}^n$.

$$X = f^1 \frac{\partial}{\partial x^1} + f^2 \frac{\partial}{\partial x^2} + \dots + f^n \frac{\partial}{\partial x^n} \qquad f^1,\dots,f^n \text{ are fncts.}$$

$$X(p) = f^1(p) \frac{\partial}{\partial x^1}\Big|_p + \dots + f^n(p) \frac{\partial}{\partial x^n}\Big|_p$$

$\boxed{\text{Prop 1.2.8}}$ $(Dx^i)\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \frac{\partial}{\partial x^j}\Big|_p (x^i) = \frac{\partial x^i}{\partial x^j}(p) = \delta_{ij}$

$$Dx^i : T_p\mathbb{R}^n \to \mathbb{R} \hookrightarrow Dx^i \in \underbrace{(T_p\mathbb{R}^n)^*}_{\text{dual space}}$$

$$V^* = L(V, \mathbb{R})$$

Def^{n}/ $d_p x^i(V_p) = V_p[x^i]$ for each $V_p \in T_p\mathbb{R}^n$ & $i \in \mathbb{N}_n$.

Def^{n}/ $d_p f(V_p) = V_p[f]$ (differential of f at p) ($d_p f \in (T_p\mathbb{R}^n)^*$

$T_p\mathbb{R}^n$ has basis $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ dual-basis

$(T_p\mathbb{R}^n)^*$ has basis $\{ d_p x^1, \dots, d_p x^n \}$ $\quad (d_p x^i)(\partial_j|_p) = \delta_{ij}$

Def^{n}/ $(T_p\mathbb{R}^n)^* = L(T_p\mathbb{R}^n, \mathbb{R})$

$$(T\mathbb{R}^n)^* = T^*\mathbb{R}^n = \bigcup_{p \in \mathbb{R}^n} \{p\} \times (T_p\mathbb{R}^n)^*$$

~~covector or dual vector fields~~ one-form

differential one-form

$$\alpha = \alpha_1 dx^1 + \dots + \alpha_n dx^n$$

$$\alpha(p) = \alpha_1(p)\, d_p x^1 + \dots + \alpha_n(p)\, d_p x^n \in (T_p\mathbb{R}^n)^*$$

$\alpha(X) = \alpha_1 f^1 + \alpha_2 f^2 + \dots + \alpha_n f^n \leftarrow$ exercise

Def^{n}/ $f : \mathrm{dom}(f) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, $p \in \mathrm{dom}(f)$, $X \in T_p\mathbb{R}^n$

$$(d_p f)(X) = X[f]$$

$\underbrace{df(\partial/\partial x^i) = \frac{\partial f}{\partial x^i}}$ btw,

$$\left.\begin{aligned} X &= X[x^1]\frac{\partial}{\partial x^1}\Big|_p + \cdots + X[x^n]\frac{\partial}{\partial x^n}\Big|_p \\ \alpha(p) &= \underline{\alpha(\partial_1|_p)}\, d_p x^1 + \cdots + \underline{\alpha(\partial_n|_p)}\, d_p x^n \end{aligned}\right\}$$ exercise

$$df = \frac{\partial f}{\partial x^1}dx^1 + \frac{\partial f}{\partial x^2}dx^2 + \cdots + \frac{\partial f}{\partial x^n}dx^n$$

$$(Df)(X)(p) = d_p f(X) = X[f].$$

Prop. 1.2.12| properties of the differential

(i) $d(fg) = (df)g + f\,dg$

(ii) $d(f+g) = df + dg$

(iii) $d(cf) = c\,df$

(iv.) $d(h \circ f) = h'(f)\,df$

fnct. on $\mathbb{R}$

$$h \circ f : \mathbb{R}^n \xrightarrow{f} \mathbb{R} \xrightarrow{h} \mathbb{R}$$

Proof| $d_p(h \circ f) = \sum_{i=1}^{n} \underbrace{\left(d_p(h\circ f)\right)\left(\frac{\partial}{\partial x^i}\Big|_p\right)}_{?} d_p x^i$

$$\begin{aligned} d_p(h \circ f)\left(\frac{\partial}{\partial x^i}\Big|_p\right) &= \frac{\partial}{\partial x^i}\Big|_p (h \circ f) \\ &= \left(\frac{\partial}{\partial x^i}[h(f(x^1,..,x^n))]\right)\Big|_p \\ &= \left(h'(f(x))\frac{\partial f}{\partial x^i}\right)\Big|_p \\ &= \underline{h'(f(p))}\frac{\partial f}{\partial x^i}(p) \end{aligned}$$

$$d_p(h \circ f) = h'(f(p)) \underbrace{\sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\Big|_p d_p x^i}_{df} = h'(f(p))\,d_p f. //$$