This is not comprehensive. However, I do hope these problems help solidify your comprehension of my notes. I also assign a few problems from Kühnel which were interesting. It is important to work through the recommended homework (note that solutions are posted on the website). I should mention, $\mathfrak{X}(S)$ is the set of smooth vector fields on S and $\Lambda^p(S)$ is the set of differential forms on S.

Problem 1 Let $\partial/\partial x^i$ and dx^i be the usual Cartesian frame and coframe. Let $Y \in \mathfrak{X}(\mathbb{R}^n)$ and $\alpha \in \Lambda^1(\mathbb{R}^n)$ show that:

$$Y = \sum_{j=1}^{n} Y[x^j]\frac{\partial}{\partial x^j} \qquad \& \qquad \alpha = \sum_{j=1}^{n} \alpha\left(\frac{\partial}{\partial x^j}\right) dx^j.$$

Problem 2 Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be smooth functions. Prove $d(fg) = (df)g + f(dg)$.

Problem 3 Let $A \in \mathbb{R}^{n\times n}$ and e_i the standard basis of $\mathbb{R}^n$. You are also given

$$Ae_1 \wedge Ae_2 \wedge \cdots \wedge Ae_n = \det(A)\, e_1 \wedge e_2 \wedge \cdots \wedge e_n.$$

If $\{f_1, \dots, f_n\} \subseteq \mathbb{R}^n$ is linearly independent then show

$$Af_1 \wedge Af_2 \wedge \cdots \wedge Af_n = \det(A)\, f_1 \wedge f_2 \wedge \cdots \wedge f_n.$$

I'd like a proof which does not assume $\det(AB) = \det(A)\det(B)$ directly, but, if it is too much trouble, I'll tolerate it.

Problem 4 Prove the graded Leibniz rule: for $\alpha \in \Lambda^p(\mathbb{R}^n)$ and $\alpha \in \Lambda^q(\mathbb{R}^n)$ we have

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

Feel free to use the multindex notation.

Problem 5 Suppose $S_{ij} = S_{ji}$ and $A_{ij} = -A_{ji}$ for all $i, j = 1, 2, \dots, n$. Prove $\displaystyle\sum_{i,j=1}^{n} S_{ij}A_{ij} = 0$.

Problem 6 Let $\alpha : I \to \mathbb{R}^n$ be a smooth parametrized curve and suppose $Y, Z \in \mathfrak{X}(\alpha)$. If $c_1, c_2 \in \mathbb{R}$ show $(c_1Y + c_2Z)'(t) = c_1Y'(t) + c_2Z'(t)$. Notice, $Y'(t), Z'(t) \in T_{\alpha(t)}\mathbb{R}^n$ so these are derivations on the set of smooth functions defined near $\alpha(t)$.

Problem 7 Let α be a non-linear regular smooth curve. Show $\tau = \dfrac{(\alpha' \times \alpha'') \bullet \alpha'''}{||\alpha' \times \alpha''||^2}$.

Problem 8 Suppose $\theta^1 = x\,dy + dz$ and $\theta^2 = (x^2 + y^2)\,dz$ and $\theta^3 = dz$. Is this a legit coframe on $\mathbb{R}^3$?

Problem 9 Consider $x = h\cosh\phi, y = h\sinh\phi, z = z$ define a system of **hyperbolic coordinates** on some subset U of $\mathbb{R}^3$. Find the frame, coframe, attitude and connection for this hyperbolic coordinate system. Define U as is convenient, yet, interesting.

Problem 10 Let $\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and define $\langle v, w \rangle = v^T \eta w$ for all $v, w \in \mathbb{R}^4$. Let us define the **pseudodistance function** $d_\eta(p,q) = \sqrt{|\langle q-p, q-p \rangle|}$. Give evidence that d_η is not a distance function and $\langle \cdot, \cdot \rangle$ is not an inner product. However, we can still study d_η preserving maps. In particular, we say $F : \mathbb{R}^4 \to \mathbb{R}^4$ is an isometry of d_η if $d_\eta(F(p), F(q)) = d_\eta(p,q)$. Work out the analog of the theorem we proved for euclidean space; that is, show every isometry of d_η is uniquely written as the in terms of a translation $T(x) = x + a$ for some $a \in \mathbb{R}^4$ and all $x \in \mathbb{R}^4$ and a **Lorentz** transformation with matrix $\Lambda \in \mathbb{R}^{4\times4}$ which satisfies $\Lambda^T \eta \Lambda = \eta$.

Hint: you may need to use a slight modification of orthonormality here, the standard basis is not orthonormal with respect to η, but, the usual formulas almost work

Problem 11 Given an orthonormal set of vector fields $E_1, \dots, E_{n-1}$ we define:

$$E_n = \sum_{j=1}^{n} \det[E_1|\cdots|E_{n-1}|U_j]U_j.$$

Show $E_n \bullet E_n = 1$. (*recall, I showed $E_i \bullet E_n = 0$ for $i = 1, \dots, n-1$ hence this completes the proof that $E_1, \dots, E_n$ so constructed is a frame in $\mathbb{R}^n$*)

Problem 12 In our study of Frenet Curves in $\mathbb{R}^n$ I implicitly used the following claim to set-up the solution to a system of 16 ODEs as if it was a system of 4 ODEs:

$$\exp(s\,M) \otimes I = \exp(s\,M \otimes I).$$

Is this claim true?

Problem 13 If $\bar{\kappa} = \kappa$ and $\bar{\tau} = -\tau$ for arclength parametrized curves $\alpha, \bar{\alpha} : I \to \mathbb{R}^3$ then α and $\bar{\alpha}$ are congruent.