

Ex: $f(x) = x^2$, $f: \mathbb{R} \rightarrow \mathbb{R}^+$

Differentiation

- Approximate a mapping by a linear mapping as closely as possible (if possible)

Definition: Let $F: U \subseteq V \rightarrow \tilde{U} \subseteq W$, then

we say F is differentiable at $x_0 \in U$

\exists a linear mapping L from $V \rightarrow W$ s.t.

$$\lim_{x \rightarrow x_0} \left(\frac{\|F(x_0 + h) - F(x_0) - L(h)\|_W}{\|h\|_V} \right) = 0$$

In this case we denote $L(h) = D_{x_0} F(h)$
or $(dF)_{x_0}(h)$ or $dF_{x_0}(h)$

Equivalently, $\exists \eta$ s.t

$$F(x) = F(x_0) + dF_{x_0}(h) + \eta$$

and $\eta \rightarrow 0$ as $x \rightarrow x_0$. (too sloppy)

Example : Let $F(x) = x$. where $x \in V$

then $dF_{x_0} = \text{id. } (dF_{x_0}(h) = h)$

$$\|F(x_0+h) - F(x_0) - dF_{x_0}(h)\| = \|x_0 + h - x_0 - h\| = 0$$

Example : $G(x) = cx + b$, then $dG_{x_0}(h) = c \cdot h$

Example : $F: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$, $F(A) = A^2$.

Find $dF_{A_0}(h)$.

$$F(A_0 + h) = (A_0 + h)^2 = A_0^2 + A_0 h + h A_0 + h^2$$

$$F(A_0) = A_0^2$$

$$\Rightarrow F(A_0 + h) - F(A_0) = A_0 h + h A_0 + h^2.$$

Conjecture : $dF_{A_0}(h) = A_0 h + h A_0$

Note $L(h) = A_0 h + h A_0$ is linear.

Yes if it is a \mathbb{R} -linear map.

$$\|F(A_0 + h) - F(A_0) - dF_{A_0}(h)\| = \|h^2\| \leq \|h\|^2$$

$$\Rightarrow 0 \leq \frac{\|F(A_0 + h) - F(A_0) - dF_{A_0}(h)\|}{\|h\|} = \frac{\|h^2\|}{\|h\|} \leq \frac{\|h\|^2}{\|h\|} = \|h\|$$

$$\lim_{h \rightarrow 0} \frac{\|F(A_0 + h) - F(A_0) - dF_{A_0}(h)\|}{\|h\|} = 0$$

Ex: $F: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$

Let $F(A) = e^A$

Conjecture: $d e^A(h) = (I + A)h$.

Proof: Homework ...

Also HW: Find the differential of $f(A) = \det(A)$
 $f(A) = \det(A)$
diff at $x_0 \implies$ Continuity at x_0

If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ then:

$$F: V \subseteq \mathbb{R}^n \rightarrow \tilde{V} \subseteq \mathbb{R}^m$$

Can be differentiated via Jacobian arguments.

More precisely, if F is diff at x_0 then:

$$dF_{x_0}: \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ which is linear.}$$

Hence by F.T.L.A, $\exists F'(x_0) \in \mathbb{R}^{m \times n}$

$$dF_{x_0}(h) = F'(x_0) h.$$

Definition: $F'(x_0)$ = Jacobian Matrix of F at x_0

Note: $F(x_0) = [dF_{x_0}]$. From M.321,

$$[dF_{x_0}] = [dF_{x_0}(e_1) | dF_{x_0}(e_2) | \dots | dF_{x_0}(e_n)]$$

We pause to discuss partial differentiation and we'll see how it connects to our general theory of diff currently discussed.

$$\text{Definition : } (D_{P_0} F)(v) = \lim_{h \rightarrow 0} \left[\frac{F(P_0 + hv) - F(P_0)}{h} \right]$$

- This is the directional derivative of F at P_0 in the v -direction.
- Unlike Calculus III, no assumption that $\|v\|=1$
- Nothing more than feeding a line

$\vec{r}(h) = P_0 + hv$ into F and you could say
 $g(h) = (F \circ \vec{r})(h)$ and

$$D_{P_0} F(v) = g'(0)$$

$$\boxed{\text{Definition : } (\delta_j F)(P_0) = (D_{P_0} F)(e_j)}$$

Partial derivative is directional derivative along a coordinate direction.

~~Remember~~ If dF_{x_0} exists then the Jacobian matrix is given in terms of partial derivatives,

$$\boxed{dF_{x_0} = \underline{F'(x_0)} = [\delta_1 F(x_0) | \dots | \delta_n F(x_0)]}$$

the differential the derivative

of course we also denote $\delta_j F = \frac{\delta F}{\delta x_j}$

Conversely: If $F'(x_0)$ exists for $F: V \subseteq \mathbb{R}^n \rightarrow \tilde{V} \subseteq \mathbb{R}^m$ then does dF_{x_0} exists in the sense that F is diff at x_0 ?

Example: No. $F(x, y) = \begin{cases} x+1 & \text{if } y=0 \\ y & \text{if } x=0 \\ 0 & \text{if } xy \neq 0 \end{cases}$

Clearly $\partial_x F(0, 0) = 1$, $\partial_y F(0, 0) = 1$
 So $F'(0, 0) = (1, 1)$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ D.N.E}$$

Example: ~~$f(x) = x^2 \sin(\frac{1}{x})$. Show $f'(0) = 0$.~~

Definition: F is continuously diff at x_0 if $\exists V \subseteq \mathbb{R}^n$ s.t
 the mapping $x \mapsto dF_x$ is continuous for each $x \in V$
 and we assume $x_0 \in V$. (Note: I assume that F is differentiable at each point in V)

Ex: $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ is differentiable at $x=0$.

You can show $f'(0) = 0$. However, $x \neq 0$,

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

$\therefore f'$ is not continuous at $x=0$.

The mapping $x \mapsto df_x = f'(x)$

Theorem: $x \mapsto df_x$ continuous iff each component of $F'(x)$ is continuous

So continuity in the abstract sense (not yet defined) is replaced w/ checking continuity of partials of component function.

Proof:

Theorem : If $F: U \subseteq \mathbb{R}^n \rightarrow \tilde{U} \subseteq \mathbb{R}^m$ is
continuous diff. at $x_0 \in U$ then F is diff at x_0 .

Proof : We're given $\partial_j F$ are continuous on some neighbourhood
 $U' \subseteq U$

$$\lim_{x \rightarrow x_0} \partial_j F(x) = \partial_j F(x_0).$$

We claim $\lim dF_{x_0}(h) = F'(x_0)h$

$$= \sum_{j=1}^n \partial_j F(x_0) h_j$$

~~$$\Rightarrow \|F(x_0+h) - F(x_0) - dF_{x_0}(h)\|$$~~

$$\leq \|F(x_0+h) - F(x_0) - \sum_{j=1}^n \partial_j F(x_0) h_j\|$$

$$h = h_1 e_1 + h_2 e_2 + \dots + h_n e_n$$

$$F(x_0+h) - F(x_0 + \underbrace{h_1 e_1}_{h_1} + \dots + \underbrace{h_n e_n}_{h_n}) + F(x_0 + h_2 e_2 + \dots + h_n e_n)$$

$$- F(x_0 + \underbrace{h_3 e_3 + \dots + h_n e_n}_{h_3}) + F(x_0 + h_3 e_3 + \dots + h_n e_n)$$

$$- F(x_0 + \underbrace{h_4 e_4 + \dots + h_n e_n}_{h_4}) + F(x_0 + h_4 e_4 + \dots + h_n e_n)$$

$$\vdots$$

$$+ F(x_0 + \underbrace{h_n e_n}_{h_n}) - F(x_0) = F(x_0 + h) - F(x_0)$$

Inverse and Implicit Function Theorem

Q1: If $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ then can we find a local inverse for F at x_0 ?

Does $\exists U \subseteq \mathbb{R}^n$ s.t $(F|_U)^{-1}$ exists?

Q2: If $G: \mathbb{R}^{n+k} \rightarrow \mathbb{R}^k$ then when can we solve $G(x,y)$

$G(x,y) = k \in \mathbb{R}^k$, for y ? I.e., when can we find

$f: \mathbb{R}^k \rightarrow \mathbb{R}^k$ s.t $G(x, f(x)) = k$

for some $x \in \text{domain}(G)$?

A1: Inverse function theorem says Yes

If $F'(x_0)$ is invertible.

$$y = F(x_0 + h) \underset{\approx}{=} F(x_0) + F'(x_0)h.$$

$$h = [y - F(x_0)] \left(F'(x_0) \right)^{-1} = F^{-1}(y)$$

A2: Yes. If $\frac{\partial G}{\partial y}$ is invertible.

$$F(x_0 + h) - F(x_0) = \sum_{j=1}^n (F(x_0 + \widehat{h}_j) - F(x_0 + \widehat{h}_{j+1}))$$

$$\text{By M.V.T } \exists x_j^* \text{ s.t } F(x_0 + \widehat{h}_j) - F(x_0 + \widehat{h}_{j+1}) = \delta_j F(x_j^*) h_j.$$

$$\Rightarrow F(x_0 + h) - F(x_0) = \sum_{j=1}^n \delta_j F(x_j^*) h_j$$

$$\Rightarrow F(x_0 + h) - F(x_0) = \sum_{j=1}^n \delta_j F(x_0) h_j$$

$$= \sum_{j=1}^h (\delta_j F(x_j^*) - \delta_j F(x_0)) h_j$$

$$\frac{\|F(x_0 + h) - F(x_0) - \sum_{j=1}^n \delta_j F(x_0) h_j\|}{\|h\|} = \frac{\|\sum_{j=1}^n (\delta_j F(x_j^*) - \delta_j F(x_0)) h_j\|}{\|h\|}$$

$$\leq \sum_{j=1}^n \|\delta_j F(x_j^*) - \delta_j F(x_0)\| \frac{\|h_j\|}{\|h\|}$$

$$\leq \sum_{j=1}^n \|\delta_j F(x_j^*) - \delta_j F(x_0)\| \ll \varepsilon.$$

∴ Done!

Chain Rule:

$$F \circ G : U \subseteq \mathbb{R}^m \xrightarrow{G} \mathbb{R}^p \xrightarrow{F} \mathbb{R}^n$$

Theorem: $d(F \circ G)_p = (dF)_{G(p)} \circ (dG)_p$ linear transformation.

$(F \circ G)'(p) = F'(G(p)) \cdot G'(p)$ where F' , G' are the Jacobian matrices of F & G respectively.

Ex: $F(x, y, z) \in \mathbb{R}$

$$G(t, s) = \langle G_1(t, s), G_2(t, s), G_3(t, s) \rangle$$

$$F : \mathbb{R}^3 \rightarrow \mathbb{R} \quad G : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$F' \in \mathbb{R}^{1 \times 3} : F' = [\partial_x F, \partial_y F, \partial_z F]$$

$$G' \in \mathbb{R}^{3 \times 2} : G' = \begin{pmatrix} \partial_t G_1 & \partial_s G_1 \\ \partial_t G_2 & \partial_s G_2 \\ \partial_t G_3 & \partial_s G_3 \end{pmatrix}$$

$$\mathbb{R}^{1 \times 2} \Rightarrow (F \circ G)' = \left[\frac{\partial}{\partial t}(F \circ G), \frac{\partial}{\partial s}(F \circ G) \right] = F' G' = \begin{bmatrix} \partial_x F & \partial_y F & \partial_z F \end{bmatrix} \begin{pmatrix} \partial_t G_1 & \partial_s G_1 \\ \partial_t G_2 & \partial_s G_2 \\ \partial_t G_3 & \partial_s G_3 \end{pmatrix}$$

$$= \left[\frac{\partial F}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial t}, \frac{\partial F}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial s} \right]$$

Newton's Method

Definition: The mapping $\varphi: [a, b] \rightarrow [a, b]$ is a contraction mapping w/ contraction constant $k < 1$ if:

$$|\varphi(x) - \varphi(y)| \leq k|x-y| \quad \forall x, y \in [a, b]$$

Theorem: φ is a contraction mapping w/ constant $k < 1$ then φ has unique fixed point x^* .

Moreover, given $x_0 \in [a, b]$ the sequence $\{x_n\}$ defined inductively by $x_{n+1} = \varphi(x_n)$ converges to x^*

In particular the distance between

$$|x_n - x^*| \leq \frac{k^n |x_0 - x_1|}{1-k}$$

Proof: $|x_{n+1} - x_n| = |\varphi(x_n) - \varphi(x_{n-1})| \leq k|x_n - x_{n-1}|$

$$\Rightarrow |x_{n+1} - x_n| \leq k^n |x_1 - x_0|$$

If $0 < n < m$ then. Let $\varepsilon > 0$. $\exists N$ s.t. $\forall n, m > N, k^n < \frac{\varepsilon(1-k)}{|x_1 - x_0|}$

$$|x_n - x_m| \leq |x_n - x_{n-1}| + \dots + |x_{m+1} - x_m|$$

$$\leq (k^n + \dots + k^{m-1}) |x_1 - x_0|$$

$$= k^n (1 + \dots + k^{m-n-1}) |x_1 - x_0|$$

$$< k^n |x_1 - x_0| < \varepsilon \Rightarrow \{x_n\} \text{ is Cauchy.}$$

$\Rightarrow \boxed{x \rightarrow x^*}$

Fixed n , we have:

$$|x_n - x_m| \leq \frac{k^n |x_1 - x_0|}{1-k}$$

$$\lim_{n \rightarrow \infty} |x_n - x_m| \leq \frac{k^n |x_1 - x_0|}{1-k}$$

$$|x_n - x^*| \leq \frac{k^n |x_1 - x_0|}{1-k}$$

φ is continuous at x^*

$$\lim_{n \rightarrow \infty} (\varphi(x_n)) = \lim_{n \rightarrow \infty} (x_{n+1}) = x^*$$

$$\Rightarrow \varphi(x^*) = x^*$$

β $\exists x^{**}$ s.t. $\varphi(x^{**}) = x^{**}$

$$\Rightarrow |\varphi(x^* - x^{**})| = |\varphi(x^*) - \varphi(x^{**})| \leq k|x^* - x^{**}|$$
$$\Rightarrow k \geq 1 \quad (!)$$

$$\Rightarrow x^* \text{ is unique } \checkmark$$

Theorem:

Consider $f: [a, b] \rightarrow \mathbb{R}$, diff w/ $f(a) < 0 < f(b)$

and $0 < m < f(x) \leq M \quad \forall x \in [a, b]$. Given

$x_0 \in [a, b]$, the sequence $\{x_n\}_0^\infty$ defined inductively by

$$x_{n+1} = x_n - \frac{f(x_n)}{M}$$

Converges to unique root $x^* \in [a, b]$ of $f(x) = 0$

$$|x_n - x_*| \leq \frac{|f(x_0)|}{m} \left(1 - \frac{m}{M}\right)^n$$

Theorem 1.3:

Let $f: \mathbb{R} \rightarrow \mathbb{R}$, $f \in C^1(\mathbb{R})$ continuity differentiable on \mathbb{R} .

s.t $f(a) = b$ and $f'(a) \neq 0$

Then $\exists U = [a-\delta, a+\delta]$ around a and $V = [b-\varepsilon, b+\varepsilon]$

s.t $\exists x_* \in V$ given any $y_* \in V$, the sequence

$\{x_n\}_0^\infty$ defined by :

$$x_0 = a, \quad x_{n+1} = x_n - \frac{f(x_n) - y_*}{f'(a)}$$

Converges to a unique point $x_* \in U$ s.t $f(x_*) = y_*$.

Proof: Choose $\delta > 0$ so small that

$$|f'(a) - f'(x)| \leq \frac{1}{2} |f'(a)| \quad \text{if } x \in U = [a-\delta, a+\delta]$$

Let $\varepsilon = \frac{1}{2} \delta |f'(a)|$. Construct $V = [b-\varepsilon, b+\varepsilon]$

We need to show

$$\varphi(x) = x - \frac{f(x) - y^*}{f'(a)}$$

is contraction map of V if $y^* \in V$.

Exercise

Show if f, g are invertible.

Theorem 1.4: Let $G: \mathbb{R}^2 \rightarrow \mathbb{R}$ be C^1 function,
 $(a, b) \in \mathbb{R}^2$ s.t $G(a, b) = 0$ and $D_2 G(a, b) \neq 0$

$(\frac{\partial G}{\partial y}(a, b) \neq 0)$. Then \exists cont. funct. $f: J \rightarrow \mathbb{R}$

defined on closed interval centered at a s.t

$y = f(x)$ solves $G(x, y) = 0$ on neighborhood of (a, b)

Moreover, $\{f_n\}_0^\infty$ a seq of func defined inductively by

$$f_0(x) = b, \quad f_{n+1}(x) = f_n(x) - \frac{G(x, f_n(x))}{\frac{\partial G}{\partial y}(a, b)}$$

Conv. uniformly to f on J

Theorem: Let $G: \mathbb{R}^{m+1} \rightarrow \mathbb{R}$ be C^1 function, and $(\vec{a}, b) \in \mathbb{R}^m \times \mathbb{R} = \mathbb{R}^{m+1}$ a point s.t $G(\vec{a}, b) = 0$ and $D_{m+1} G(\vec{a}, b) \neq 0$.

Then \exists a continuous function $f: J \rightarrow \mathbb{R}$, defined on a closed cube $J \subset \mathbb{R}^m$ centered at $\vec{a} \in \mathbb{R}^m$, s.t $y = f(\vec{x})$

solves the equation $G(\vec{x}, y) = 0$ in a nhood of (\vec{a}, b)

In particular, if the function $\{f_n\}_0^\infty$ are defined inductively by

$$f_0(\vec{x}) = b, \quad f_{n+1}(\vec{x}) = f_n(\vec{x}) - \frac{G(\vec{x}, f_n(\vec{x}))}{D_{m+1} G(\vec{a}, b)}$$

then the sequence $\{f_n\}_0^\infty$ converges uniformly to f on J .

Given $C \subseteq \mathbb{R}^n$ the map $\varphi: C \rightarrow C$ is contraction mapping w/ constant if $\|\varphi(x) - \varphi(y)\|_0 \leq k \|x - y\|_0$.

If $k < 1$, then we obtain a fixed p.t theorem:

$$\|v\| = \max \{\|v_1\|, \|v_2\|, \dots, \|v_n\|\}$$

Theorem: If a c.m w/ $k < 1$ and C is closed, bounded subset of \mathbb{R}^m . Then φ has unique fixed point x^* ; $\varphi(x^*) = x^*$

Moreover, $x_{m+1} = \varphi(x_m) \rightarrow x^*$

$$\text{and } \|\vec{x}_m - \vec{x}_*\|_0 \leq k^m \frac{\|\vec{x}_0 - \vec{x}_1\|_0}{1-k}$$

Intuition: $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and df_a is invertible.

$$y = f(a + \vec{x}) = f(a) + f'(a)(\vec{x})$$

$$\Rightarrow \vec{x} = (f'(a))^{-1}[y - f(a)] + a = f^{-1}(y)$$

Lemma 3.2: Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be C' map s.t $f(0) = 0$ and $df_0 = I$. S also $\|df_x - I\| \leq \varepsilon < 1 \quad \forall x \in C_r =$

$$C_r = \{x \in \mathbb{R}^n \mid \|x\|_0 \leq r\}$$

Then $C_{(1-\varepsilon)r} \subset f(C_r) \subset C_{(1+\varepsilon)r}$.

Moreover, if $V = \text{int}(C_{(1-\varepsilon)r})$ and $U = \text{int}(C_r \cap f^{-1}(V))$ then $f: U \rightarrow V$ is 1-1 & onto mapping w/ inverse g which is diff at zero.

Finally, the local inverse map $g: V \rightarrow U$ is the limit of a seq of successive approximation, $\{g_m\}_0^\infty$ defined inductively on V by

$$g_0(y) = 0, g_{m+1}(y) = g_m(y) - f(g_m(y))^{-1}y.$$

Theorem: $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$

C^1 in nbhd of a (call it W)

$$f'(a) \in GL(n)$$

Then f is locally invertible,

\exists nbhd $U \subset W$ of a and V of $b = f(a)$ and $a \mapsto$
 C^1 mapping $g: V \rightarrow W$ s.t:

$$g(f(x)) = x \quad \text{for } x \in U$$

$$f(g(y)) = y \quad \text{for } y \in V$$

In particular, the loc. inv. g is the limit of

$\{g_k\}_0^\infty$ defined by

$$g_0(0) = a, g_{k+1}(y) = g_k(y) - f'(a)^{-1}[f(g_k(y)) - y]$$

for $y \in V$