Lecture Notes for Differential Geometry

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introduction and motivations for these notes

Certainly many excellent texts on differential geometry are available these days. These notes most closely echo Barrett O'neill's classic *Elementary Differential Geometry* revised second edition. I taught this course once before from O'neil's text and we found it was very easy to follow, however, I will diverge from his presentation in several notable ways this summer.

1. I intend to use modern notation for vector fields. I will resist the use of **bold** characters as I find it frustrating when doing board work or hand-written homework.

2. I will make some general n-dimensional comments here and there. So, there will be two tracks in these notes: first, the direct extension of typical American third semester calculus in $\mathbb{R}^3$ (with the scary manifold-theoretic notation) and second, some results and thoughts for the n-dimensional context.

I hope to borrow some of the wisdom of Wolfgang Kühnel's *Differential Geometry: Curves-Surfaces-Manifolds*. I think the purely three dimensional results are readily acessible to anyone who has taken third semester calculus. On the other hand, general n-dimensional results probably make more sense if you've had a good exposure to abstract linear algebra.

I do not expect the student has seen advanced calculus before studying these notes. However, on the other hand, I will defer proofs of certain claims to our course in advanced calculus.

Chapter 1

introduction

These notes largely concern the geometry of curves and surfaces in $\mathbb{R}^n$. I try to use a relatively modern notation which should allow the interested student a smooth[1] transition to further study of abstract manifold theory. That said, most of what I do in this chapter is merely to dress multivariate analysis in a new notation.

In particular, I decided to sacrifice the pedagogy of O'neill's text in part here; I try to introduce notations which are not transitional in nature. For example, I introduce coordinates with **contravariant** indices and we adopt the **derivation** formulation of tangent vectors early in our discussion. The tangent bundle and space are openly discussed. However, we do not digress too far into bundles.

The purpose of this chapter is primarily notational, if you had advanced calculus then it would be almost entirely a review. However, if you have not had advanced calculus then take some comfort that most of the missing details here are provided in that course.

1.1 points and vectors

We define $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ for $n = 1, 2, 3, \ldots$. A point $p \in \mathbb{R}^n$ has cartesian coordinates $p^1, p^2, \ldots, p^n$. These are labels, not powers, in our notation. Addition and scalar multiplication are:

$$(p+q)^i = p^i + q^i \qquad \& \qquad (cp)^i = cp^i$$

for $i \in \mathbb{N}_n$. Notice, in the context of $\mathbb{R}^n$ if we take two points p and q then $p+q$ and cp are once more in $\mathbb{R}^n$. However, if the space we considered was a solid sphere then you might worry that the sum of two points landed outside. This space $\mathbb{R}^n$ is essentially our mathematical universe for the most part in this course. We study curves, surfaces and manifolds[2] and many of the calculations we make are reasonable since these curves, surfaces and manifolds are sets of points in $\mathbb{R}^n$ (often $n = 3$ for this course).

Define[3] $(e_i)^j = \delta_{ij}$ and note $p \in \mathbb{R}^n$ is formed by a linear combination of this **standard basis**:

$$p = (p^1, p^2, \ldots, p^n) = p^1(1, 0, \ldots, 0) + \cdots + p^n(0, \ldots, 1) = p^1 e_1 + \cdots p^n e_n.$$

[1] pun partly intended

[2] ok, if all goes well, we'll see some examples of manifolds which are not in $\mathbb{R}^n$, but, for now...

[3] in case you forgot, $\delta_{ij} = 0$ if $i \neq j$ and is 1 if $i = j$.

Of course, we can either envision p as the point with Cartesian coordinates $p^1, \dots, p^n$, or, we can envision p as a vector eminating from the origin out to the point p. However, this identification is only reasonably because of the unique role $(0, 0, \dots, 0)$ plays in $\mathbb{R}^n$. When we attach vectors to points other than the origin we really should take care to denote the attachment explicitly.

Definition 1.1.1. *tangent space and the tangent bundle.*

We define $T_p\mathbb{R}^n = \{p\} \times \mathbb{R}^n$ to be the **tangent space** to p of $\mathbb{R}^n$. The **tangent bundle** of $\mathbb{R}^n$ is defined by $T\mathbb{R}^n = \cup_{p \in \mathbb{R}^n} \{p\} \times T_p\mathbb{R}^n$.

Conceptually, $T_p\mathbb{R}^n$ is the set of vectors attached or based at p and the tangent bundle is the collection of all such vectors at all points in $\mathbb{R}^n$. By a slight abuse of notation, a typical element of $T_p\mathbb{R}^n$ has the form (p, v) where p is the **point of attachment** or **base-point** and v is the **vector part** of (p, v).

The set $T_p\mathbb{R}^n$ is naturally a vector space by:

$$(p, v) + (p, w) = (p, v + w) \qquad \& \qquad c(p, v) = (p, cv).$$

Moreover, the **dot-product** and **norm** are defined by the usual formulas on the vector part:

$$(p, v) \bullet (p, w) = v \bullet w \qquad \& \qquad ||\,(p, v)\,|| = ||v|| = \sqrt{v \bullet v}.$$

We say $(p, v), (p, w) \in T_p\mathbb{R}^n$ are **orthogonal** when $(p, v) \bullet (p, w) = v \bullet w = 0$. Given $S \subseteq T_p\mathbb{R}^n$ we may form $S^\perp = \{(p, v) \mid (p, v) \bullet (p, s) = 0 \text{ for all } (p, s) \in S\}$. When S is also a subspace of the tangent space at p we have $T_p\mathbb{R}^n = S \oplus S^\perp$. Furthermore, if S plays the role of the **tangent space** to some object at the point p then $S^\perp$ is identified as the **normal space** at p. We see the size of the normal space varies according to the ambient $\mathbb{R}^n$. For example, a curve C has a one-dimensional tangent space and the normal space has dimension $n - 1$. In $\mathbb{R}^2$ you have a normal line to C, in $\mathbb{R}^3$ there is a normal plane to C, in $\mathbb{R}^4$ there is a normal 3-volume to C. Or, for a surface S with a two-dimensional tangent plane, we have a normal line for S in $\mathbb{R}^3$, or a normal plane for S in $\mathbb{R}^4$. Much of what is special to $\mathbb{R}^3$ depends directly on the fact that the normal space to a line is a plane and the normal space to a plane is a line. This duality is implicitly used in many steps.

If we assign a vector to each point along $S \subseteq \mathbb{R}^n$ then we say such assignment is a **vector field** on S. A vector field on S naturally corresponds to a function $X : S \to T\mathbb{R}^3$ such that $X(p) \in \{p\} \times T_p\mathbb{R}^n$ for each $p \in S$. There is a natural mapping $\pi : T\mathbb{R}^n \to \mathbb{R}^n$ defined by $\pi(p, v) = p$ for all $(p, v) \in T\mathbb{R}^n$. Notice, to say $X : S \to T\mathbb{R}^n$ is a vector field on S is to say $\pi(X(p)) = p$ for each $p \in S$. Or, simply $\pi \circ X = id_S$. In the usual langauge of bundles we say X is a section of $T\mathbb{R}^n$ over S. Again, keeping with the theme of generality in this section, S could be a curve, surface or higher-dimensional manifold. In each case, we can use a section of the tangent bundle to encapsulate what it means to have a vector field on that object.

In these notes, for the sake of matrix calculations, I consider $\mathbb{R}^n = \mathbb{R}^{n \times 1}$ (column vectors). This means we can use standard linear algebra when faced with questions about $T_p\mathbb{R}^n$. We simply remove the p on (p, v) and apply the usual theory. In the next section we find a new notation for (p, v) which brings new insight and is standard in modern treatments of manifold theory.

1.2 on tangent and cotangent spaces and bundles

Before we define the directional derivative on $\mathbb{R}^n$ we pause to define **coordinate functions**.

Definition 1.2.1. *coordinate functions*

Let $x^i(p) = p^i$ for $i = 1, 2, \dots, n$. In $n = 3$, $x(p) = p^1$, $y(p) = p^2$ and $z(p) = p^3$. In other words, we reserve notation x^i to denote a function from $\mathbb{R}^n$ to $\mathbb{R}$ defined by $x^i(p) = p^i$.

I will try to reserve the notation x, y, z for $\mathbb{R}^3$ and $x^1, \dots, x^n$ for $\mathbb{R}^n$ to denote **functions**. In contrast, p, q typically denote some fixed, but arbitrary, point. For example:

Example 1.2.2. *Define* $f = 2x^1 - 3x^4 + (x_n)^2$ *then* $f(p) = \left(2x^1 - 3x^4 + (x_n)^2\right)(p)$ *and by the usual addition of functions* $f(p) = 2p^1 - 3p^4 + (p_n)^2$.

Example 1.2.3. *Define* $f = x + y^2 + z^3$. *Observe* $f(a, b, c) = a + b^2 + c^3$.

The directional derivative measures the rate of change in a given function f, at a given point p, in a given direction v. Traditionally, in third-semester American calculus, we assume the given direction vector is a **unit-vector**, but, that is merely a convenience of exposition. We make no such assumption in what follows.

Definition 1.2.4. *directional derivative*

Suppose $f : dom(f) \subseteq \mathbb{R}^n \to \mathbb{R}$ is a smooth function and $p \in dom(f)$. The **directional derivative** of f with respect to $(p, v) \in T_p\mathbb{R}^n$ is denoted $(Df)(v)(p)$ and defined by:

$$(Df)(v)(p) = \lim_{t \to 0} \frac{f(p + tv) - f(p)}{t}.$$

Notice $\alpha(t) = p + tv$ is a line which passes through p at $t = 0$ and has direction vector v. We can rephrase the definition in terms of a derivative of f composed with the parametrized line α:

$$(Df)(v)(p) = \lim_{t \to 0} \frac{(f \circ \alpha)(t) - (f \circ \alpha)(0)}{t} = (f \circ \alpha)'(0).$$

But, we also know the chain-rule for multivariate functions, and as we assume f is smooth we obtain the following refinement of the directional derivative through partial derivatives of f:

Proposition 1.2.5. *directional derivative by partial derivatives.*

Suppose $f : dom(f) \subseteq \mathbb{R}^n \to \mathbb{R}$ is a smooth function and $p \in dom(f)$. The **directional derivative** of $(p, v) \in T_p\mathbb{R}^n$ is given by:

$$(Df)(v)(p) = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(p).$$

Proof: if $x^1, \dots, x^n$ are functions of t then the chain-rule tells us

$$\frac{d}{dt} f(x^1, \dots, x^n) = \frac{\partial f}{\partial x^1}\frac{dx^1}{dt} + \cdots + \frac{\partial f}{\partial x^n}\frac{dx^n}{dt}$$

However, as $x^i = p^i + tv^i$ we have $\frac{dx^i}{dt} = v^i$ and the proposition follows. $\square$

If I am teaching an audience which is scared by limits, then the natural definition to give is simply:

$$(Df)(v)(p) = (\nabla f)(p) \bullet v.$$

In practice, it's usually easier to use the above formula as opposed to direct calculation of the limit. But, the limit is important as it shows us the foundational concept; $(Df)(v)(p)$ desribes the rate of change in f at p in the v direction.

Let $x^1, \dots, x^n$ denote the Cartesian coordinate functions of $\mathbb{R}^n$ then denote

$$\frac{\partial}{\partial x^i}\bigg|_p (f) = \frac{\partial f}{\partial x^i}(p)$$

for each $p \in \mathbb{R}^n$. Also, we use $\frac{\partial}{\partial x^i}\big|_p = \partial_i|_p$ when there is no danger of confusion. Notice there is a bijective correspondance:

$$(p, (v^1, \dots, v^n)) \leftrightarrows v^1 \frac{\partial}{\partial x^1}\bigg|_p + \cdots + v^n \frac{\partial}{\partial x^n}\bigg|_p$$

It is customary to view $T_p\mathbb{R}^n$ as the span of the coordinate derivations:

Definition 1.2.6. *tangent space as a set of derivations.*

$$T_p\mathbb{R}^n = \{v^1\partial_1|p + \cdots + v^n\partial_n|p \mid (v^1, \dots, v^n) \in \mathbb{R}^n\}.$$

Furthermore, the notation (p, v) is often replaced with v_p. The beauty of this notation is in part the following truth: the directional derivative of v_p on f is simply given by v_p acting on f.

$$(Df)(v_p) = v_p[f].$$

Let us pause to state a few properties of derivations which should be familiar

Proposition 1.2.7. *properties of derivations*

If $X = v^1\partial_1|p + \cdots + v^n\partial_n|p$ and f, g smooth functions at p and c a constant then

$$X[fg] = X[f]g(p) + f(p)X[g], \quad X[f+g] = X[f] + X[g], \quad X[cf] = cX[f].$$

Proof: these all follow from the form of X and the properties of partial derivatives at a point. $\square$

In another context, you'd likely use the above properties to **define** a derivation as the properties make no reference to coordinates.

Vector fields are also naturally covered in this notation: if X is a vector field on $S \subseteq \mathbb{R}^n$ then we have component functions $f^1, \dots, f^n$ on S for which

$$X(p) = f^1(p)\frac{\partial}{\partial x^1}\bigg|_p + \cdots + f^n(p)\frac{\partial}{\partial x^n}\bigg|_p$$

for each $p \in S$. However, as this holds for each p we can express X as

$$X = f^1\frac{\partial}{\partial x^1} + \cdots + f^n\frac{\partial}{\partial x^n}.$$

Proposition 1.2.8. *on the directional derivatives of coordinate functions.*

Let x^i be the i-th coordinate function on $\mathbb{R}^n$. Then, $(Dx^i)\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \delta_{ij}$.

Proof: follows from our reformulation of directional derivatives:

$$(Dx^i)\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \frac{\partial}{\partial x^j}\Big|_p (x^i) = \frac{\partial x^i}{\partial x^j} = \delta_{ij}. \qquad \square$$

Traditionally, we replace Dx^i with dx^i. In particular, let us define:

Definition 1.2.9. *coordinate differentials*

Define $d_p x^i(v_p) = v_p[x^i]$ for each $v_p \in T_p\mathbb{R}^n$ and $i \in \mathbb{N}_n$.

Since $T_p\mathbb{R}^n$ is a vector space with basis $\{\partial/\partial x^i|_p\}_{i=1}^n$ it is natural to seek out the **dual basis** for $(T_p\mathbb{R}^n)^* = \{\alpha : T_p\mathbb{R}^n \to \mathbb{R} \mid \alpha \in L(T_p\mathbb{R}^n, \mathbb{R})\}$. The dual space $(T_p\mathbb{R}^n)^*$ is called **cotangent space**. Indeed, it should be clear from the discussion above that we already have the dual-basis in hand:

$$\{d_p x^1, d_p x^2, \dots, d_p x^n\}$$

is a basis for $(T_p\mathbb{R}^n)^*$ which is dual to the basis $\{\partial_1|p, \partial_2|p, \cdots \partial_n|p\}$ of $T_p\mathbb{R}^n$. There is also a **cotangent bundle** of $\mathbb{R}^n$ which is formed by the disjoint union of all the cotangent spaces:

Definition 1.2.10. *cotangent space and the cotangent bundle.*

Cotangent space at p for $\mathbb{R}^n$ is the set of $\mathbb{R}$-valued linear functions on $T_p\mathbb{R}^n$. The **cotangent bundle** of $\mathbb{R}^n$ is defined by $T^*\mathbb{R}^n = \cup_{p\in\mathbb{R}^n}\{p\} \times (T_p\mathbb{R}^n)^*$

A section over S of the cotangent bundle gives us an assignment of a covector at p for each $p \in S$. You might expect we would call such a section a **covector field**, however, it is customary to call such an object a **differential one-form**. A **one-form** α on S is a covector-valued function on S for which $\alpha(p) \in (T_p\mathbb{R}^n)^*$ for each $p \in S$. For example, $dx^1, dx^2, \dots, dx^n$ (with the natural interpretation $dx^i(p) = d_p x^i$ for all $p \in \mathbb{R}^n$) are one-forms on $\mathbb{R}^n$.

It is useful to know how we can isolate the component functions of a vector field or one-form by evaluation of the appropriate object. Both of these results[4] flow from the identities $dx^i(\partial_j) = \delta_{ij}$ and $\partial_i x^j = \delta_{ij}$. If Y is a vector field on $\mathbb{R}^n$ and α is a differential one-form on $\mathbb{R}^n$ then

$$Y = Y^1\partial_1 + \cdots Y^n\partial_n \qquad \& \qquad \alpha = \alpha_1 dx^1 + \cdots + \alpha_n dx^n$$

where the functions Y^i and α_i are given by:

$$Y^i = Y[x^i] \qquad \& \qquad \alpha_i = \alpha(\partial_i).$$

In third semester American calculus we typically use the differential notation in a formal sense. We are now in the position to expose the deeper reasons for such notation. The **total differential** is a differential one-form in our current study.

[4]you might find these as homework exercises

Definition 1.2.11. *the differential of a real-valued function on $\mathbb{R}^n$*

Suppose $f : dom(f) \subseteq \mathbb{R}^n \to \mathbb{R}$ is a smooth function and $p \in dom(f)$. The **differential** of f at p is denoted $d_p f$. We define $d_p f \in (T_p\mathbb{R}^n)^*$ by

$$d_p f(Y) = Y[f]$$

for each $Y \in T_p\mathbb{R}^n$. The assignment $p \mapsto d_p f$ gives the differential one-form df on $\mathbb{R}^n$.

Of course, $df(\partial/\partial x^i) = \frac{\partial f}{\partial x^i}$ hence we find that

$$df = \frac{\partial f}{\partial x^1}dx^1 + \frac{\partial f}{\partial x^2}dx^2 + \cdots + \frac{\partial f}{\partial x^n}dx^n.$$

Thus, for $X \in T_p\mathbb{R}^n$ we arrive at the following formula for the directional derivative of f at p in the X direction:

$$\boxed{(Df)(X)(p) = d_p f(X) = X[f].} \tag{1.1}$$

If we had a vector field X then $df(X)$ is a function. Likewise, an arbitrary one-form α and a vector field X can be combined to form a function $\alpha(X)$.

The following properties of the differential are not difficult to prove:

Proposition 1.2.12. *on the directional derivatives of coordinate functions.*

Suppose f, g are functions from $\mathbb{R}^n$ to $\mathbb{R}$ and h is a function on $\mathbb{R}$ then

(i.) $d(fg) = (df)g + f(dg)$,

(ii.) $d(f+g) = df + dg$,

(iii.) $d(cf) = cdf$,

(iv.) $d(h \circ f) = h'(f)df$.

Proof: I leave the first three to the reader. We prove (iv.). Consider $h : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$ and recall the chain-rule:

$$\frac{\partial}{\partial x^i}\left[h(f(x))\right] = h'(f(x))\frac{\partial f}{\partial x^i} \quad \Rightarrow \quad d(h \circ f)(x^i) = h'(f(x))\frac{\partial f}{\partial x^i}.$$

Thus, $d(h \circ f) = \sum_{i=1}^{n} h'(f(x))\frac{\partial f}{\partial x^i}dx^i = h'(f(x))\sum_{i=1}^{n}\frac{\partial f}{\partial x^i}dx^i = h'(f(x))df.$ $\square$

Identity (iv.) frees us from our roots.

Example 1.2.13. *Let* $f = \sqrt{(x^1)^2 + \cdots + (x^n)^2}$ *clearly* $f^2 = (x^1)^2 + \cdots + (x^n)^2$ *and thus* $d(f^2) = 2fdf$ *and* $d(f^2) = 2x^1dx^1 + \cdots + 2x^ndx^n$ *thus*

$$df = \frac{x^1dx^1 + \cdots + x^ndx^n}{\sqrt{(x^1)^2 + \cdots + (x^n)^2}}.$$

1.3 the wedge product and differential forms

We saw in the last section the directional derivative naturally leads us to define the differential of a function. In particular, we saw that a function f was converted to a one-form df by the operation d. To continue this story we need to introduce the **wedge product**. The wedge product generalizes the cross product of three dimensional vector algebra. Or, perhaps it would be more accurate to say, the wedge product allows us a computational device to implement determinants without explicitly writing determinants. The combination of the wedge product with total differentiation brings us to the **exterior derivative**. It is the combination of exterior differentiation and the algebra of the wedge product which makes for many elegant simplifications in what would otherwise require much more sophisticated calculations. I will admit, it is probably a bit strange at the beginning, but, if we stick with it then we'll gain skills which allow us to do calculus in one of the most general settings.

I'll define the wedge product formally by it's properties[5]. The wedge takes p**-form** and "wedges" it with a q**-form** to make a $(p+q)$-form. If we take a sum of terms with k differentials wedged together then this gives us a k-**form**. A 0-form is a function. A 1-form is an expression of the form $\alpha = \sum_i \alpha_i dx^i$ where α_i are functions. A 2-form can be written as $\beta = \sum_{i,j} \frac{1}{2}\beta_{ij} dx^i \wedge dx^j$ where β_{ij} are functions. A k-form can be written $\gamma = \sum_{i_1,\dots,i_k} \frac{1}{k!}\gamma_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ where $\gamma_{i_1,\dots,i_k}$ are functions. The essential properties of the wedge product are as follows:

(i.) $\wedge$ is an **associative** product; $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.

(ii.) $\wedge$ is **anticommutative** on differentials; $dx^i \wedge dx^j = -dx^j \wedge dx^i$

(iii.) $\wedge$ has a **distributive** property; $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$.

It's not much work to show $\wedge$ is anticommutative on arbitary one-forms. If $\alpha = \sum_i \alpha_i dx^i$ and $\beta = \sum_j \beta_j dx^j$ are one forms then, by the above properties:

$$\begin{aligned}
\alpha \wedge \beta &= \left(\sum_i \alpha_i dx^i\right) \wedge \left(\sum_j \beta_j dx^j\right) \\
&= \sum_i \sum_j \alpha_i \beta_j dx^i \wedge dx^j \\
&= -\sum_j \sum_i \beta_j \alpha_i dx^j \wedge dx^i \\
&= -\left(\sum_j \beta_j dx^j\right)\left(\sum_i \alpha_i dx^i\right) \\
&= -\beta \wedge \alpha.
\end{aligned}$$

Up to this point I have indicated our differential forms have coefficients which are functions. If we were to follow terminology similar to that for vectors then we would rather call differential forms something like form-fields. However, this is not usually done. Naturally, if we wished to work at a single point then the functions mentioned above would be traded for sets of constants[6].

[5] there are elegant ways in terms of abstract algebra, or the concrete multilinear mapping discussion you can see in my advanced calculus notes, here, I just say how it works

[6] we could discuss the algebra of wedge products of a given vector space V without any discussion of differentials, but, I mostly keep our focus on the objects we use for the calculational core of this course. The wedge product algebra would provide another way to capture linear independence. For example, $v \wedge w = 0$ iff v is linearly dependent on w. The same holds for a k-fold wedge product. In contrast, determinants worked only for n-vectors in $\mathbb{R}^n$.

When we take the wedge product we create an object of a new type, hence, it is an **exterior product**[7]; from α a p and β a q form we obtain $\alpha \wedge \beta$ a $p+q$ form. What's to stop us from making p and q infinitely large? It's tempting to think these go on forever. However, notice, $dx^i \wedge dx^i = -dx^i \wedge dx^i$ implies $dx^i \wedge dx^i = 0$. Therefore, the only way to have a nontrivial form is to build it from the wedge product of **distinct** differentials. In $\mathbb{R}^n$ there are at most n-distinct differentials. The form $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ is called the **top-form** as there is not other form which beats it by degree. The determinant is linked to this top-form in a natural manner. Let us define $\wedge$ on column vectors in the same fashion as described here for differentials. Then, the determinant is implicit defined by the wedge product of the columns of the matrix:

$$Ae_1 \wedge Ae_2 \wedge \cdots \wedge Ae_n = \det(A) e_1 \wedge e_2 \wedge \cdots \wedge e_n.$$

This identity equally well applies if we replace the standard basis with a set of linearly independent vectors[8]. For example, if B is invertible then $Be_1, \dots, Be_n$ are linearly independent and:

$$ABe_1 \wedge ABe_2 \wedge \cdots \wedge ABe_n = \det(A) Be_1 \wedge Be_2 \wedge \cdots \wedge Be_n.$$

But, then

$$Be_1 \wedge Be_2 \wedge \cdots \wedge Be_n = \det(B) e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

and we find

$$ABe_1 \wedge ABe_2 \wedge \cdots \wedge ABe_n = \det(A)\det(B) e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

from which we find $\det(AB) = \det(A)\det(B)$. In the case A or B was not invertible the set of vectors $ABe_1, ABe_2, \dots, ABe_n$ is linearly dependent and we can easily argue the wedge product of such vectors is zero hence $\det(AB) = 0$ in such a case. What's my point? This is powerful algebra. If you recall the trouble of proving the product rule of determinants in linear algebra then this should help you appreciate why the wedge product is worth our time.

The wedge product is **graded commutative** for forms of homogeneous degree.

Proposition 1.3.1. *graded commutativity of the wedge product.*

If α is a form built from a sum of terms each with k-differentials, and likewise β is a l-form then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

Proof: I'll take this opportunity to introduce multi-index notation. We write $\alpha = \sum_{I \in \mathcal{I}_k} \alpha_I dx^I$ and $\beta = \sum_{J \in \mathcal{I}_l} \beta_J dx^J$ where $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $dx^J = dx^{j_1} \wedge \cdots \wedge dx^{j_l}$ and $\mathcal{I}_k$ is the set of k-tuples of distinctly increasing indices in $\mathbb{N}_n$ (assume we're working in $\mathbb{R}^n$). Calculate:

$$\alpha \wedge \beta = \left[\sum_{I \in \mathcal{I}_k} \alpha_I dx^I \right] \wedge \left[\sum_{J \in \mathcal{I}_l} \beta_J dx^J \right] = \sum_{J \in \mathcal{I}_l} \sum_{I \in \mathcal{I}_k} \beta_J \alpha_I \epsilon dx^J \wedge dx^I \quad \star$$

[7]in contrast, a binary operation takes two objects in the set and returns another element in the set. In some sense, the term "exterior" is just short-sighted, the wedge product does act within the set of all differential forms over $\mathbb{R}^n$ and does provide an honest binary operation on that larger set of forms of all possible nontrivial degrees

[8]this seems like an interesting homework question

where $\epsilon = \pm 1, 0$. In particular $\epsilon = 0$ if any differential is repeated. It is ± 1 depending on how we have to move dx^I past dx^J. Consider, as J has l-differentials we generate $(-1)^l$ as we shift a differential through dx^J:

$$\begin{aligned} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^J &= (-1)^l dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}} \wedge dx^J \wedge dx^{i_k} \\ &= (-1)^l (-1)^l dx^{i_1} \wedge \cdots \wedge dx^{i_{k-2}} \wedge dx^J \wedge dx^{i_{k-1}} \wedge dx^{i_k} \\ &\;\vdots \\ &= \underbrace{(-1)^l (-1)^l \cdots (-1)^l}_{k-copies} dx^J \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \end{aligned}$$

But, this means $\epsilon = (-1)^{kl}$ for the nontrivial terms and the identity then follows from $\star$. $\square$

This means forms of even degree commute with all other forms.

Let us return to differentiation. The **exterior derivative** of a one-form $\alpha = \sum_i \alpha_i dx^i$ is defined by:

$$d\alpha = \sum_i d\alpha_i \wedge dx^i.$$

where $d\alpha_i = \sum_j (\partial_j \alpha_i) dx^j$ for each i. To calculate the exterior derivative we simply take the total differential of the component functions of the given form and wedge that with the differentials that come attached to the form at the outset. Let me give a general definition for our reference:

Definition 1.3.2. *exterior derivative.*

For a k-form $\gamma = \sum_{i_1,\dots,i_k} \frac{1}{k!}\gamma_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ we define,

$$d\gamma = \sum_{i_1,\dots,i_k,j} \frac{1}{k!} \frac{\partial \gamma_{i_1,\dots,i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

The exterior derivative of the k-form is a $(k+1)$-form. There is an elegant, coordinate free, manner in which we can define the exterior derivative. The proof of coordinate independence in our current notation is involved. But, we leave that for another course. There is a graded-product rule for the exterior derivative. You can show:

Proposition 1.3.3. *graded-Leibniz rule for the wedge product of differential forms.*

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta.$$

where I use $|\alpha|$ to denote the **degree** of α. In particular, if α is a k-form then $|\alpha| = k$.

Proof: not too hard. Really much like the one I already provided to prove Proposition 1.3.1. $\square$

In the case $\alpha = f$ and $\beta = g$ we have two zero-forms and the relation above reduces to the usual product rule for differentials

$$d(fg) = (df)g + f(dg)$$

If α is a one-form and β is a k-form then

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta - \alpha \wedge d\beta \qquad \& \qquad d(\beta \wedge \alpha) = (d\beta) \wedge \alpha + (-1)^{|\beta|}\beta \wedge (d\alpha).$$

It's possibly a good exercise to check the consistency of the relations above by several applications of the graded commutativity relation. The identity below is important:

$$d(d\gamma) = 0$$

or, simply $d^2 = 0$ on arbitrary differential forms on $\mathbb{R}^n$. The proof rests primarily on the fact that partial differentiation with respect to x^j and x^m commutes for nice functions. Recall,

$$d\gamma = \sum_{i_1,\dots,i_k,j} \frac{1}{k!}\frac{\partial \gamma_{i_1,\dots,i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Hence, applying the definition of exterior derivative once more:

$$d(d\gamma) = \sum_{i_1,\dots,i_k,j,m} \frac{1}{k!}\frac{\partial^2 \gamma_{i_1,\dots,i_k}}{\partial x^m \partial x^j} dx^m \wedge dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} = 0$$

I claim[9] the sum works out to zero since the sum has partial derivatives which are symmetric in j, m summed against $dx^m \wedge dx^j = -dx^j \wedge dx^m$.

The example below might make a little more sense after you read the subsection on three-dimensional vector calculus written in terms of forms and exterior derivatives. I leave these here as they expand on the importance of the $d^2 = 0$ identity.

Example 1.3.4. *A force $\vec{F}$ is conservative if there exists f such that $\vec{F} = -\nabla\phi$. In the langauge of differential forms, this means the one-form $\omega_{\vec{F}}$ represents a conservative force if $\omega_{\vec{F}} = \omega_{-\nabla\phi} = -d\phi$. Observe, $\omega_{\vec{F}} = -d\phi$ implies $d\omega_{\vec{F}} = -d^2\phi = 0$. As an application, consider $\omega_{\vec{F}} = -ydx + xdy + dz$, is $\vec{F}$ conservative ? Calculate:*

$$d\omega_{\vec{F}} = -dy \wedge dx + dx \wedge dy + d(dz) = 2dx \wedge dy \neq 0$$

thus $\vec{F}$ is not conservative.

Example 1.3.5. *It turns out that Maxwell's equations can be expressed in terms of the exterior derivative of the potential one-form A. The one-form contains both the voltage function and the magnetic vector-potential from which the time-derivative and gradient derive the electric and magnetic fields. In spacetime the relation between the potentials and fields is simply $F = dA$. The choice of A is far from unique. There is a* **gauge freedom**. *In particular, we can add an exterior derivative function of spacetime λ and create $A' = A + d\lambda$. Note, $dA' = dA + d^2\lambda = dA$ hence A and A' generate the same electric and magnetic fields (which make up the Faraday tensor F)*

The identity $d^2 = 0$ essentially captures the necessity of partial derivatives commuting. However, it does this without explicit reference of coordinates and across forms of arbitrary degree.

[9] I usually assign this as an advanced calculus homework

1.3.1 the exterior algebra in three dimensions

Let f, P, Q, R, A, B, C and G be functions on some subset of $\mathbb{R}^3$ then

(i.) f is a 0-form

(ii.) $\alpha = Pdx + Qdy + Rdz$ is a 1-form

(iii.) $\beta = A\,dy \wedge dz + B\,dz \wedge dx + C\,dx \wedge dy$ is a 2-form

(iv.) $\gamma = G\,dx \wedge dy \wedge dz$ is a 3-form

We would like to connect to the familiar world of vector fields. Therefore, we introduce the **work-form-mapping** and the **flux-form-mapping**.

Definition 1.3.6. *work and flux form correspondance*

Let $\vec{F} = \langle P, Q, R\rangle$ be a vector field on some subset of $\mathbb{R}^3$ then we define

$$\omega_{\vec{F}} = P\,dx + Q\,dy + R\,dz \qquad \& \qquad \Phi_{\vec{F}} = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$$

We recover the cross-product and triple product in terms of wedge products of the above mappings.

Proposition 1.3.7. *vector algebra in differential form.*

Let $\vec{A}, \vec{B}, \vec{C}$ be vectors in $\mathbb{R}^3$ and $c \in \mathbb{R}$

(i.) $\omega_{\vec{A}} \wedge \omega_{\vec{B}} = \Phi_{\vec{A}\times\vec{B}}$.

(ii.) $\omega_{\vec{A}} \wedge \omega_{\vec{B}} \wedge \omega_{\vec{C}} = \vec{A} \bullet (\vec{B} \times \vec{C})\, dx \wedge dy \wedge dz$

(iii.) $\omega_{\vec{A}} \wedge \Phi_{\vec{B}} = (\,\vec{A} \bullet \vec{B}\,)\, dx \wedge dy \wedge dz$.

Proof: left to reader. $\square$

The differential calculus of vector fields involves the gradient, curl and divergence. All three of these are generated in the framework of the exterior derivative.

Proposition 1.3.8. *differential vector calculus in differential form.*

Let $f, \vec{F} = \langle F_x, F_y, F_\rangle\rangle, \vec{G} = \langle G_x, G_y, G_z\rangle$ be differentiable in $\mathbb{R}^3$

(i.) $df = \omega_{\nabla f} = (\partial_x f)\, dx + (\partial_y f)\, dy + (\partial_z f)\, dz$

(ii.) $d\omega_{\vec{F}} = \Phi_{\nabla\times\vec{F}} = (\partial_y F_z - \partial_z F_y)\, dy \wedge dz + (\partial_z F_x - \partial_x F_z)\, dz \wedge dx + (\partial_x F_y - \partial_y F_x)\, dx \wedge dy$,

(iii.) $d\Phi_{\vec{G}} = (\,\nabla \bullet \vec{G}\,)\, dx \wedge dy \wedge dz = (\,\partial_x G_x + \partial_y G_y + \partial_z G_z\,)\, dx \wedge dy \wedge dz$

Proof: left to reader. $\square$

It is interesting to note how $d^2 = 0$ recovers several useful vector calculus identities. For example,

$$d(df) = d\omega_{\nabla f} = \Phi_{\nabla\times\nabla f} = 0 \quad \Rightarrow \quad \nabla \times \nabla f = 0.$$

or,

$$d(d\omega_{\vec{F}}) = d\Phi_{\nabla\times\vec{F}} = \nabla \bullet (\nabla \times \vec{F})\, dx \wedge dy \wedge dz = 0 \quad \Rightarrow \quad \nabla \bullet (\nabla \times \vec{F}) = 0.$$

The graded Leibniz rule also reveals several interesting product rules. For example,

$$\begin{aligned} d(\omega_{\vec{A}} \wedge \omega_{\vec{B}}) &= d\omega_{\vec{A}} \wedge \omega_{\vec{B}} - \omega_{\vec{A}} \wedge d\omega_{\vec{B}} \\ &= \Phi_{\nabla\times\vec{A}} \wedge \omega_{\vec{B}} - \omega_{\vec{A}} \wedge \Phi_{\nabla\times\vec{B}} \\ &= \Big((\nabla \times \vec{A}) \bullet \vec{B} - \vec{A} \bullet (\nabla \times \vec{B}) \Big)\, dx \wedge dy \wedge dz. \end{aligned}$$

But, we also know $\omega_{\vec{A}} \wedge \omega_{\vec{B}} = \Phi_{\vec{A}\times\vec{B}}$ hence

$$d(\omega_{\vec{A}} \wedge \omega_{\vec{B}}) = d\Phi_{\vec{A}\times\vec{B}} = \nabla \bullet (\vec{A} \times \vec{B})\, dx \wedge dy \wedge dz.$$

Therefore, $\nabla \bullet (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \bullet \vec{B} - \vec{A} \bullet (\nabla \times \vec{B})$. I can derive this with Levi-Civita calculations without much trouble, but, I wonder, can you ? The point I intend to make here: the wedge product automatically builds all manner of complicated vector identities into a few simple, generalizable, algebraic operations. The beauty of differential forms in revealing the true structure of integral vector calculus is no less shocking. But, I leave it for another time. In fact, I leave many things for another time here. I merely hope we've said enough to make our use of the wedge product and exterior calculus a bit less bizarre in the remainder of the course.

1.4 paths and curves

If you read O'neill very carefully, you'll notice he distinguishes between *curve* and *Curve*[10]. I suppose there are many choices of terms to make here. I think Kühnel's was clean and nicely descriptive so I'll use his terminology as our default.

Definition 1.4.1. *parametrized curve and its curve.*

A smooth parametrized curve in $\mathbb{R}^n$ is a smooth function $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^n$ where I is an open interval. The point-set $\alpha(I) \subseteq \mathbb{R}^n$ is a smooth curve.

The term "smooth" could be replaced with a number of other adjectives. For example, we could talk about *continuous* or *once-differentiable* or *continuously differentiable* or *k-times differentiable* curves. In our study of curves in the next chapter we study smooth curves which have non-vanishing velocity; that is, *regular curves*. The assumption of smoothness for functions and curves is probably too greedy for most of our purposes, I merely impose it for the sake of being certain I can take as many derivatives as we wish. For example, one derivative is all I need for the following:

Definition 1.4.2. *velocity of a parametrized curve.*

Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^n$ be a smooth parametrized curve. We define $\alpha'(t) \in T_{\alpha(t)}\mathbb{R}^n$ by

$$\alpha'(t) = \frac{d\alpha^1}{dt}\frac{\partial}{\partial x^1}\bigg|_{\alpha(t)} + \frac{d\alpha^2}{dt}\frac{\partial}{\partial x^2}\bigg|_{\alpha(t)} + \cdots + \frac{d\alpha^n}{dt}\frac{\partial}{\partial x^n}\bigg|_{\alpha(t)}.$$

If $\alpha'(t) \neq 0$ for all $t \in I$ then we say α is a **regular curve**.

[10] see page 21

On pages 7-8 of Kühnel discusses why regularity is a good requirement for us to make for the curves we wish to study. Of course, we could use less structure, but then we'd face pathological examples such as parametrized curves whose curve fill a rectangle in the plane, or differentiable curves which have infinitely many corners. The criteria of regularity keeps these odd features outside our study. From a big-picture perspective, regularity is a *full-rank* condition since the highest dimension possible for the tangent space to a curve is one and regularity demands the rank of the tangent space be everywhere maximal on the curve. We soon discuss similar conditions for mappings in the final section of this chapter.

The velocity of a parametrized curve is an **operator** on functions defined near the curve. Consider:

$$\begin{aligned}\alpha'(t)[f] &= \left(\frac{d\alpha^1}{dt}\frac{\partial}{\partial x^1}\bigg|_{\alpha(t)} + \frac{d\alpha^2}{dt}\frac{\partial}{\partial x^2}\bigg|_{\alpha(t)} + \cdots + \frac{d\alpha^n}{dt}\frac{\partial}{\partial x^n}\bigg|_{\alpha(t)} \right)[f] \\ &= \frac{d\alpha^1}{dt}\frac{\partial f}{\partial x^1}(\alpha(t)) + \frac{d\alpha^2}{dt}\frac{\partial f}{\partial x^2}(\alpha(t)) + \cdots + \frac{d\alpha^n}{dt}\frac{\partial f}{\partial x^n}(\alpha(t)) \\ &= \frac{d}{dt}\left[f(\alpha(t))\right]\end{aligned}$$

where in the last step we used the chain-rule for $f \circ \alpha$. The velocity $\alpha'(t)$ acts on f to tell us how f changes along α at $\alpha(t)$. We record this result for future reference:

Proposition 1.4.3. *change of function along curve.*

Suppose α is a parametrized curve and f is a smooth function defined near $\alpha(t)$ for some $t \in \mathrm{dom}(\alpha)$ then $\alpha'(t)[f] = \dfrac{d(f \circ \alpha)}{dt}$.

Example 1.4.4. *Let $\alpha(t) = p + t(q-p)$ for a given pair of distinct points $p, q \in \mathbb{R}^n$. You should identify α as the line connecting point $p = \alpha(0)$ and $q = \alpha(1)$. If we define $v = q - p$ then the velocity of α is given by:*

$$\alpha'(t) = v^1 \frac{\partial}{\partial x^1}\bigg|_{\alpha(t)} + v^2 \frac{\partial}{\partial x^2}\bigg|_{\alpha(t)} + \cdots + v^n \frac{\partial}{\partial x^n}\bigg|_{\alpha(t)}$$

Specializing to $n = 2$ and $v = \langle a, b\rangle$ we have $\alpha(t) = (p^1 + ta,\, p^2 + tb)$ and

$$\alpha'(t) = a\frac{\partial}{\partial x}\bigg|_{\alpha(t)} + b\frac{\partial}{\partial y}\bigg|_{\alpha(t)}$$

Let $f(x, y) = x^2 + y^2$ then

$$\alpha'(t)[f] = \left(a\frac{\partial}{\partial x}\bigg|_{\alpha(t)} + b\frac{\partial}{\partial y}\bigg|_{\alpha(t)} \right)[x^2 + y^2] = (2xa + 2yb)\big|_{\alpha(t)} = 2(p^1 + ta)a + 2(p^2 + tb)b.$$

As an easy to check case, take $p = (0, 0)$ hence $p^1 = 0$ and $p^2 = 0$ hence $\alpha'(t)[f] = 2t(a^2 + b^2)$. For $t > 0$ we see f is increasing as we travel away from the origin along the line $\alpha(t)$. But, f is just the distance from the origin squared so the rate of change is quite reasonable. If we were to impose $a^2 + b^2 = 1$ then t represents the distance from the origin and the result reduces to $\alpha'(t)[f] = 2t$ which makes sense as $f(\alpha(t)) = (ta)^2 + (tb)^2 = t^2(a^2 + b^2) = t^2$.

Notice that $\alpha'(t)[f]$ gives the usual third-semester-American calculus directional derivative in the direction of $\alpha'(t)$ only if we choose a parameter t for which $||\alpha'(t)|| = 1$. This choice of parametrization is known as the *arclength* or *unit-speed* parametrization.

Example 1.4.5. *Let $R, m > 0$ be constants and $\alpha(t) = (R\cos t, R\sin t, mt)$ for $t \in \mathbb{R}$. We say α is a* **helix** *with slope m and radius R. Notice $\alpha(t)$ falls on the cylinder $x^2 + y^2 = R^2$. Of course, we could define helices around other circular cylinders. The velocity vector field for α is given by:*

$$\alpha'(t) = \left(-R\sin t \frac{\partial}{\partial x} + R\cos t \frac{\partial}{\partial y} + m\frac{\partial}{\partial z} \right)\Bigg|_{\alpha(t)}$$

Then, $f(x, y, z) = x^2 + y^2$ has

$$\alpha'(t)[f] = (-2xR\sin t + 2yR\cos t)\big|_{\alpha(t)} = -2R^2\cos t\sin t + 2R^2\sin t\cos t = 0.$$

This is in good agreement with Proposition 1.4.3 as $f(\alpha(t)) = R^2$ is constant. On the other hand, $g(x, y, z) = z$ gives $\alpha'(t)[g] = m$ which shows m is proportional to the rate at which the helix rises in z. To obtain the absolute rate we would need to derive the arclength-parametrization of α.

A given curve has infinitely many parametrizations.

Definition 1.4.6. *reparametrization of curve.*

Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^n$ be a parametrized curve. If $h : J \to I$ is a smooth function on an open interval J then $\beta = \alpha \circ h : J \to \mathbb{R}^n$ is a parametrized curve which we call the **reparametrization of α by h**.

The definition above is a bit more general than I usually give in third-semester American calculus in the sense that the reparametrization need not cover the same curve. It could be that the curve of the reparametrization is just a subset of the curve of the original parametrized curve. Also, we did not assume h is injective which means β might not share the same orientation as α.

Proposition 1.4.7. *velocity of reparametrized curve*

If α is a parametrized curve $\beta = \alpha \circ h$ is a reparametrization of α by h then

$$\beta'(s) = \frac{dh}{ds}\,\alpha'(h(s))$$

Proof: by definition,

$$\beta'(s) = \frac{d\beta^1}{ds}\frac{\partial}{\partial x^1}\bigg|_{\beta(s)} + \frac{d\beta^2}{ds}\frac{\partial}{\partial x^2}\bigg|_{\beta(s)} + \cdots + \frac{d\beta^n}{ds}\frac{\partial}{\partial x^n}\bigg|_{\beta(s)}$$

But, $\beta(s) = \alpha(h(s))$ hence $\frac{d\beta^j}{ds} = \frac{d}{ds}\alpha^j(h(s)) = \frac{d\alpha^j}{ds}(h(s))\frac{dh}{ds}$ for each j and we find a factor of $\frac{dh}{ds}$ on each term which when factored out yields:

$$\beta'(s) = \frac{dh}{ds}\left[\frac{d(\alpha\circ h)^1}{ds}\frac{\partial}{\partial x^1}\bigg|_{\alpha(h(s))} + \frac{d(\alpha\circ h)^2}{ds}\frac{\partial}{\partial x^1}\bigg|_{\alpha(h(s))} + \cdots + \frac{d(\alpha\circ h)^1}{ds}\frac{\partial}{\partial x^n}\bigg|_{\alpha(h(s))} \right]$$

the proposition follows immediately as the term in square-brackets is precisely $\alpha'(h(s))$. $\square$

Honestly, the theorem above is not new. We also had this theorem in multivariate calculus. The new thing is merely the notation for expressing vectors attached to a point as derivations. I include the proof here merely to show how we work with such notation.

Example 1.4.8. *Consider the helix defined by $R, m > 0$ and $\alpha(t) = (R\cos t, R\sin t, mt)$ for $t \in \mathbb{R}$. The speed of this helix is simply $||\alpha'(t)|| = \sqrt{R^2+m^2}$. Let $h(s) = s/\sqrt{R^2+m^2}$ then if β is α reparametrized by h we calculate by Proposition 1.4.7,*

$$\beta'(s) = \frac{dh}{ds}\,\alpha'(h(s)) = \frac{1}{\sqrt{R^2+m^2}}\left(-R\sin h(s)\frac{\partial}{\partial x} + R\cos h(s)\frac{\partial}{\partial y} + m\frac{\partial}{\partial z}\right)\Big|_{\alpha(h(s))}$$

Then $||\beta'(s)|| = \frac{1}{\sqrt{R^2+m^2}}\sqrt{R^2+m^2} = 1$. Let $g(x,y,z) = z$ as in Example 1.4.5 and calculate $\beta'(s)[g] = \dfrac{m}{\sqrt{R^2+m^2}}$. If $\triangle t = 2\pi$ then the helix goes once around the z-axis. Correspondingly, $\triangle s = 2\pi\sqrt{R^2+m^2}$ and thus the change in z over a complete cycle is $\frac{m}{\sqrt{R^2+m^2}}\cdot 2\pi\sqrt{R^2+m^2} = 2\pi m$.

Definition 1.4.9. *arclength parametrization of regular parametrized curve*

If $\alpha : I \to \mathbb{R}^n$ is a regular parametrized curve and $t_o \in I$ then we define the **arclength function based at** t_o by

$$s(t) = \int_{t_o}^{t} ||\alpha'(\tau)||d\tau$$

Let h be the inverse function of the arclength function; $h(s(t)) = t$ for all $t \in I$ then define the **arclength parameterization of** α **based at** $\alpha(t_o)$ to be $\beta = \alpha \circ h$.

Suppose I is a finite interval. The image of the arclength function is the total arclength of α and that gives the domain of h in the definition. The arclength parameterization of a regular curve will share the same *orientation* as the parameterized curve α. Furthermore, the arclength parametrization is unique up to the choice of base-point. If we have two different base-points then the arclength parameterizations just differ by a simple translation: $\beta_1(s) = \beta_2(s + s_o)$. I invite the reader to prove[11] that $||\beta'(s)|| = 1$.

The nearly unique nature of the arclength parameterization of a parametrized regular smooth curve makes the arclength parameterization a good candidate for forming definitions. However, in practice, it may be difficult or even impossible to calculate $s(t)$ in terms of elementary functions. Moreover, even if we can nicely calculate the arclength function there is still no guarantee a reasonable formula for h exists. Notice, I don't cast doubt on the existence of the arclength function or its inverse, merely the existence of a **nice formula**. This claim is necessarily nebulous as we have never defined **nice**. That said, I know a nice formula when I see one. One attempt at capturing this idea is given by Risch's Algorithm (as explained in this Wikipedia article linked here)

1.5 the push-forward or differential of a map

Consider $F : \text{dom}(F) \subseteq \mathbb{R}^n \to \mathbb{R}^m$. We have $F = (F^1, \dots, F^m)$ where $F^i : \text{dom}(F) \subseteq \mathbb{R}$ are the **component functions** of F. The **Jacobian matrix** of F is an $m \times n$ matrix denoted J_F of partial derivatives of the component functions:

$$J_F = \begin{bmatrix} \partial_1 F^1 & \partial_2 F^1 & \cdots & \partial_n F^1 \\ \partial_1 F^2 & \partial_2 F^2 & \cdots & \partial_n F^2 \\ \vdots & \vdots & \cdots & \vdots \\ \partial_1 F^m & \partial_2 F^m & \cdots & \partial_n F^m \end{bmatrix}$$

[11]Notice $\frac{ds}{ds} = 1$ is not a complete proof, although, it is evidence our notation is good. See page 53 of O'neill.

there are two natural ways to parse the above:

$$J_F = [\partial_1 F|\partial_2 F|\cdots|\partial_n F] = \begin{bmatrix} \nabla F^1 \\ \hline \nabla F^2 \\ \hline \vdots \\ \hline \nabla F^m \end{bmatrix}$$

When a function is differentiable then the Jacobian matrix allows us to approximate the function by its affinization:

$$F(a+h) \approx L(a+h) = F(a) + J_F(a)h.$$

In a bit more detail,

$$F^j(a+h) \approx L^j(a+h) = F^j(a) + (\nabla F^j)(a) \bullet h$$

Sometimes we focus on the change in F at $x = a$ denoted $\triangle F$ we have:

$$\triangle F^j \approx (\nabla F^j)(a) \bullet h$$

Of course we recognize this as the directional derivative of F^j at a in the h-direction. This difference between our discussion in § 1.2 and our current context: we have to consider the change in each component function. But, besides that necessary complication, the story is the same as for functions from $\mathbb{R}^n \to \mathbb{R}$. Well, at least from the viewpoint of approximation theory. When we consider the differential then there is a bit more to say.

The change is not really what we're after. Rather, we want the infinitesimal change. To be precise, we wish to describe how tangent vectors in the domain are mapped to tangent vectors in the domain. In the one-dimensional case, the answer was given by $d_pf(X_p) = X_p(f)$. This was a number, or, if we allow p to vary we obtained a **real-valued** function $df(X)$. As we consider the problem of how tangents transform for $F : \mathbb{R}^n \to \mathbb{R}^m$ we need more structure than a number. As a motivational exercise, we'll follow O'neill and see how curves are mapped under a mapping like F. Then, the chain-rule will show us how the tangents to the curves are transformed in kind. Once that study is complete, we'll extract the proper definition in terms of the **derivation-formulation of a tangent vector**. Forgive me as I revert to third-semester American calculus-style vector notation as we motivate the **push-forward**.

Example 1.5.1. *Let $F(x^1, x^2) = (x^1 + x^2, x^1 - x^2)$. Consider a parametrized curve $\alpha(t) = (a(t), b(t))$. The image of α under F is:*

$$(F \circ \alpha)(t) = F(a(t), b(t)) = (a(t) + b(t),\ a(t) - b(t))$$

The velocity vector for α is $\alpha'(t) = \langle a'(t), b'(t)\rangle$

$$(F \circ \alpha)'(t) = \big\langle a'(t) + b'(t),\ a'(t) - b'(t)\big\rangle = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a'(t) \\ b'(t) \end{bmatrix}$$

of course, the matrix above is just the Jacobian matrix of F. In particular, we see a tangent vector $\langle 1, 0\rangle$ would be moved to $\langle 1, 1\rangle$ and the tangent vector $\langle 0, 1\rangle$ is transported to $\langle 1, -1\rangle$. The columns of the Jacobian matrix tell us how the basis tangent vectors are transformed. Here we took for granted the existence of a cartesian coordinate system in the range. To allow for different dimensions, we ought to make our observation in a more generalizable fashion: I'll use e_1, e_2 for the domain and f_1, f_2 for the range:

$$e_1 \mapsto f_1 + f_2 \qquad \& \qquad e_2 \mapsto f_1 - f_2.$$

Now, in the derivation notation for tangent vectors, $e_1 = \frac{\partial}{\partial x^1}, e_2 = \frac{\partial}{\partial x^2}$ *and for cartesian coordinates* (y^1, y^2) *for the range* $f_1 = \frac{\partial}{\partial y^1}, f_2 = \frac{\partial}{\partial y^2}$. *We have:*

$$\frac{\partial}{\partial x^1} \mapsto \frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2} \qquad \& \qquad \frac{\partial}{\partial x^2} \mapsto \frac{\partial}{\partial y^1} - \frac{\partial}{\partial y^2}.$$

Example 1.5.2. *Another example,* $F(x^1, x^2) = (e^{x^1+x^2}, \sin x^2, \cos x^2)$. *Once more, consider the curve* $\alpha = (a,b)$ *hence* $\alpha' = \langle a', b'\rangle$ *and*

$$(F \circ \alpha)' = \langle e^{a+b}(a'+b'),\ (\cos b)b',\ (-\sin b)b'\rangle$$

We find the tangent $\langle a', b'\rangle = \langle 1, 0\rangle$ *maps to* $\langle e^{a+b},\ 0,\ 0\rangle$ *whereas the tangent* $\langle a', b'\rangle = \langle 0, 1\rangle$ *maps to* $\langle e^{a+b},\ \cos b,\ -\sin b\rangle$ *with respect to the point* $\alpha = (a,b)$ *of the curve. The Jacobian of* F *at* (a,b) *is:*

$$J_F(a,b) = \begin{bmatrix} e^{a+b} & e^{a+b} \\ 0 & \cos b \\ 0 & -\sin b \end{bmatrix}.$$

Following the notation of the last example, but now with (y^1, y^2, y^3) *coordinates for the image,*

$$\frac{\partial}{\partial x^1} \mapsto e^{a+b}\frac{\partial}{\partial y^1} \qquad \& \qquad \frac{\partial}{\partial x^2} \mapsto e^{a+b}\frac{\partial}{\partial y^1} + \cos b\frac{\partial}{\partial y^2} - \sin b\frac{\partial}{\partial y^3}.$$

Notice, $y^j \circ F = F^j$ *is immediate from the definition of cartesian coordinates and component functions. What we really have above is:*

$$\frac{\partial}{\partial x^1} \mapsto \sum_{j=1}^{3} \frac{\partial(y^j \circ F)}{\partial x^1}\frac{\partial}{\partial y^j} \qquad \& \qquad \frac{\partial}{\partial x^2} \mapsto \sum_{j=1}^{3} \frac{\partial(y^j \circ F)}{\partial x^2}\frac{\partial}{\partial y^j}$$

But, even the above is not quite accurate as it does not indicate the true point-dependence. Annoyingly, I must write:

$$\frac{\partial}{\partial x^1}\bigg|_p \mapsto \sum_{j=1}^{3} \frac{\partial(y^j \circ F)}{\partial x^1}(p)\frac{\partial}{\partial y^j}\bigg|_{F(p)} \qquad \& \qquad \frac{\partial}{\partial x^2}\bigg|_p \mapsto \sum_{j=1}^{3} \frac{\partial(y^j \circ F)}{\partial x^2}(p)\frac{\partial}{\partial y^j}\bigg|_{F(p)}$$

Of course, including the point-dependence on the cartesian coordinate derivations in overkill. However, later when we deal with curved coordinate systems the coordinate derivations will aquire a point-dependence like $\hat{r}$ *or* $\hat{\theta}$ *(discussed in my multivariate callculus notes).*

From § 1.2 recall that we may write $\frac{\partial F^j}{\partial x^i}(p) = d_pF^j\left(\frac{\partial}{\partial x^i}\big|_p\right)$. Thus, recognizing $F^j = y^j \circ F$ we find the result of our last motivating example can be written:

$$\frac{\partial}{\partial x^i}\Big|_p \ \mapsto\ \sum_{j=1}^{3} d_pF^j\left(\frac{\partial}{\partial x^i}\Big|_p\right)\frac{\partial}{\partial y^j}\bigg|_{F(p)}.$$

Extending this linearly brings us to see why the next definition is made:

Definition 1.5.3. *differential of a mapping or, the push-forward*

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function at p and suppose $X \in T_p\mathbb{R}^n$ and $y^1, \dots, y^m$ are Cartesian coordinates such that $F^j = y^j \circ F$ for each $j \in \mathbb{N}_m$ then we define the **differential of** F at p to be the mapping from $T_p\mathbb{R}^n$ to $T_{F(p)}\mathbb{R}^m$ defined by:

$$d_pF(X) = \sum_{j=1}^{m} d_pF^j\,(X)\,\frac{\partial}{\partial y^j}\bigg|_{F(p)}.$$

Alternatively, we may denote $d_pF(X) = F_{*p}(X)$. Or, is we wish to think of p as varying then $p \mapsto d_pF$ may be denoted F_* or dF.

It is also useful to view the above in slightly different terms:

$$d_pF(X) = \sum_{j=1}^{m} X[F^j]\frac{\partial}{\partial y^j}\bigg|_{F(p)}.$$

If we permit the identification of the formula of F with the coordinates in the image, that is we set $F^j = y^j$ then the formula simplifies further:

$$d_pF\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \sum_{j=1}^{m} \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}\bigg|_{F(p)}.$$

Perhaps you saw such a calculation in your multivariate calculus course. For example, the problem of writing $\partial/\partial r$ in terms of derivatives with respect to x, y.

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}.$$

This is the push-forward of $\partial/\partial r$ with respect to the map $F(r,\theta) = (r\cos\theta, r\sin\theta)$ where the domain and range of F are viewed as the same plane. Thus, we observe the push-forward is sometimes understood as a coordinate change. However, this is a special case as generally the domain and codomain need not be the same set, or even the same dimension.

The reason F_* is called the **push-forward** is that it pushes a vector field in the domain of F to another vector field in the range of F. Well, not so fast, the previous sentence is only true if F does not misbehave. For example, if F thrice wraps a circle in the domain around an oval in the image then we might have several vectors mapped to a given point on the oval. A vector field is an assignment of one vector to each point, so, $F_*(X)$ would not be a vector field in such a case. On the other hand, if F is injective locally then we can expect to map vector fields to vector fields by push-forward of the restriction of F.

In the advanced calculus course we show that local injectivity for a continuously differentiable function $F : \mathbb{R}^n \to \mathbb{R}^m$ is implied by the injectivity of the differential of the function at a point. In particular, we define F to be **regular** at p if the differential d_pF is full-rank. If $F : \mathbb{R}^n \to \mathbb{R}^m$ then d_pF has full-rank at p if $J_F(p)$ has n-linearly-independent columns. The **inverse function theorem** states that when F is a continuously differentiable function which is regular at p then there exist $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ for which $F|_U : U \to V$ is invertible with continuously differentiable inverse. In the case $n = m$ we can check the regularity of F at p by showin $\det J_F(p) \neq 0$. This theorem is a **local** result in the sense that it does not imply the invertibilty of F for its whole domain.

For example, the polar coordinate transformation is locally invertible away from the origin, but, it always faces the 2π-angle degeneracy if we have a domain which includes a circle around the origin.

The other theorem we need at times from advanced calculus is the **implicit function theorem**. If $F : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable with n-linearly-independent final columns in J_F at (q, p) then we can find a continuously differentiable function $G : \mathbb{R}^n \to \mathbb{R}^r$ such that $q = G(p)$ and $y = G(x)$ solves $F(x, y) = c$ for (x, y) near (q, p). I think this is a bit harder to understand. Let me just give two examples which illustrate a typical use of the theorem to express curves and surfaces as **graphs**.

Example 1.5.4. *Let $F(x, y) = x^2 + y^2$ then $F(x, y) = R^2$ is a circle and*

$$J_F = \begin{bmatrix} 2x & 2y \end{bmatrix}$$

we see $y \neq 0$ implies the last column is nonzero hence we may solve for y near such points. In this case, $G(x) = \pm\sqrt{R^2 - x^2}$ where we choose $\pm$ appropriate to the location of the local solution.

Example 1.5.5. *Let $F(x, y, z) = \cos(x) + y + z^2$ then*

$$J_F = \begin{bmatrix} -\sin(x) & 1 & 2z \end{bmatrix}$$

this tells me I can solve for $z = z(x, y)$ when $z \neq 0$, or I can solve for $y = y(x, z)$ anywhere on $F(x, y, z) = c$, or I can solve for $x = x(y, z)$ when $x \neq n\pi$ for $n \in \mathbb{Z}$. Notice we can rearrange coordinates to put x or y as the last coordinate.

I conclude this section with a few comments which ought to be made somewhere. You can skip them in the first reading of these notes.

The definition I offer above is actually not the definition given in O'neill. Rather, O'neill defines F_{*p} as the mapping which takes tangents at p with velocity v to tangents of $F(p + tv)$ at $F(p)$. Then he derives from that the result:

$$F_*(\alpha') = \beta'$$

where $\beta = F \circ \alpha$. In fact, the above result is sometimes used to define the differential. This definition is elegant and has certain advantages over my coordinate-based definition. The reason for the different defnitions is simply that we have freedom to view tangent vectors in differential geometry in several formalisms. To be careful, if I was to use the curve definition, I would use an **equivalence class** of curves and write:

$$d_pF([\alpha']) = [(F \circ \alpha)']$$

You can define an isomorphism between equivalence classes of curves and derivations of smooth functions at a given point. Then, if you translate O'neill's curve-based definition through that isomorphism then you'll obtain the definition I gave in this section. Both formulations have their merit and O'neill has a way of using both without being explicit. The explicit nature of what I say here ruins the art of the presentation. My apologies.

Remark 1.5.6.

The push-forward causes a notational problem in the one-dimensional case. We defined $d_pf(X) = X[f]$ for $f : \text{dom}(f) \subseteq \mathbb{R}^n \to \mathbb{R}$ and $p \in \text{dom}(f)$ with $X \in T_p\mathbb{R}^n$. But, if $x^1, \dots, x^n$ and t are the coordinates on $\mathbb{R}^n$ and $\mathbb{R}$ respectively then our definition of the push-forward implies:

$$f_{*p}(X) = X[f] \frac{\partial}{\partial t}\bigg|_{f(p)}$$

But, this formula does not quite match our definition of the differential. The usual remedy is to identify $\frac{\partial}{\partial t}\Big|_{f(p)}$ with 1. We should keep this convention in mind as we occasionally need to use it.

Chapter 2

curves and frames

We begin with a brief section on metric and normed spaces. Then we define the essential vector algebra of dot and cross products. Orthogonality for $T_p\mathbb{R}^3$ is described. Ideally much of this is a review for those who have take linear algebra and multivariate calculus. The dot product is used to select components with respect to orthonormal bases and the cross product is used to create vectors which are orthogonal to a given pair of vectors.

We define a **frame** to be a set of vector fields which are orthonormal at each point. We show that the formulas which derive from a given frame are identical to those which are known for the standard Cartesian frame. The frame must be positively oriented to maintain the usual cross product. We study the standard examples of the cylindrical and spherical frames. Arguably all of this ought to be shown in the standard multivariate calculus course as the use of frames is rather common in applications of vector calculus.

Curves and the calculus of vector fields along a curve are studied. We explain how the *Frenet frame* may be attached to regular nonlinear parametrized curves in $\mathbb{R}^3$. We then derive the Frenet Serret equations which show how the tangent, normal and binormal vector fields evolve along a curve in response to nontrivial *curvature* or *torsion*. We prove a selection of standard theorems about lines, circles and planar curves. The interested reader may consult Kühnel's *Differential Geometry: Curves-Surfaces-Manifolds* to read about Frenet curves and the theory of multiple curvatures and many other theorems about curves we will not cover in this chapter.

We lay a foundation of investigation which is continued in further chapters. The *covariant derivative* is introduced and its basic properties are derived for $\mathbb{R}^3$. Then, the study of the covariant derivative with respect a frame brings us to consider connection form which generalizes the curvature and torsion we saw earlier for the Frenet frame. Fortunately, the *attitude matrix* of a frame allows us calculate the connection form with a simple synthesis of linear algebra and exterior calculus. Finally, we introduce *coframes* consisting of three differential one-forms which are dual to the given frame of vector fields. We derive by Cartan's equations of structure by an efficient combination of matrix notation and exterior calculus. We return to this construction in later chapters, it is given here mostly to make the connection to the Frenet frame clear to the reader. Moreover, Cartan's Structure Equations are found in many contexts beyond this course. Indeed, the so-called **tetrad** formulation of general relativity is built over this sort of calculus. I hope this introduction to frames in $\mathbb{R}^3$ is easy to follow and ideally we build some intuition for the method of frames. Note: I begin to use O'neill's notation U_1, U_2, U_3 for $\partial_x, \partial_y, \partial_z$ in this chapter.

2.1 on distance in three dimensions

Let me briefly review the concept of distance in $\mathbb{R}^3$. If $p, q \in \mathbb{R}^3$ then recall $p \bullet q = p^1q^1 + p^2q^2 + p^3q^3$. We find the distance from the origin to p is $\sqrt{p \bullet p} = \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2}$. Given a pair of points p, q we find the distance between the points by calculating the length of the line-segment $\overline{pq} = q - p$:

$$d(p,q) = \sqrt{(q^1 - p^1)^2 + (q^2 - p^2)^2 + (q^3 - p^3)^2}.$$

In fact, $\mathbb{R}^3$ paired with $d : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is a **metric space** as d satisfies the following properties:

(i.) **symmetric**: $d(p,q) = d(q,p)$ for all $p, q \in \mathbb{R}^3$

(ii.) **distinct points are at nontrivial distance**: $d(p,q) = 0$ iff $p = q$.

(iii.) **non-negative**: $d(p,q) \geq 0$ for all $p, q \in \mathbb{R}^3$

(iv.) **triangle inequality**: $d(p,r) \leq d(p,q) + d(q,r)$ for all $p, q, r \in \mathbb{R}^3$

We say $B_\epsilon(x_o) = \{p \in \mathbb{R}^3 \mid d(p, x_o) < \epsilon\}$ is an **open ball** of radius ϵ centered at x_o. A point $p \in \mathbb{R}^3$ is an **interior point** of $U \subseteq \mathbb{R}^3$ if $p \in U$ and there exists $\epsilon > 0$ for which $B_\epsilon(p) \subseteq U$. If each point in $U \subseteq \mathbb{R}^3$ is an interior point then we say U is an **open set**. For example, you can show open balls are open sets. A set V is said to be a **closed set** if $\mathbb{R}^3 - V$ is an open set. If we take an open set U and restrict d to $U \times U$ then you can verify that d is also defines metric space structure on U. However, the concept of an **inner product space** is not so forgiving.

An[1] **inner product space** is a vector space V paired with an inner product $\langle , \rangle : V \times V \to \mathbb{R}$ which must satisfy:

(i.) **bilinearity**: for all $x, y, z \in V$ and $c \in \mathbb{R}$:

$$\langle cx + y, z \rangle = c\langle x, z \rangle + \langle y, z \rangle \qquad \text{and} \qquad \langle x, cy + z \rangle = c\langle x, y \rangle + \langle x, z \rangle.$$

(ii.) **symmetric**: $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$

(iii.) **positive definite**: $\langle x, x \rangle \geq 0$ for all $x \in V$ and $\langle x, x \rangle = 0$ iff $x = 0$.

It is simple to verify $\langle x, y \rangle = x \bullet y$ defines an inner product on $\mathbb{R}^3$. A normed vector space is similarly defined. We say a real vector space V has norm $|| \cdot || : V \to [0, \infty)$ when the function $|| \cdot ||$ satisfies the following properties:

(i.) **positive definite**: $||x|| \geq 0$ for all $x \in V$ and $||x|| = 0$ iff $x = 0$.

(ii.) for each $x \in V$ and $c \in \mathbb{R}$, $||cx|| = |c|\,||x||$ where $|c| = \sqrt{c^2}$.

(iii.) **triangle inequality**: $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in V$

Notice $\mathbb{R}^3$ is a normed linear space with respect to the norm $||x|| = \sqrt{x \bullet x}$.

Sometimes a normed vector space is called a normed linear space. Sequences can be studied in normed linear spaces in the usual manner: $x_n : \mathbb{N} \to V$ is a **converges** to $x_o \in V$ if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ for which $n > N$ implies $||x_n - x_o|| < \epsilon$. Likewise, $\{x_n\}$ is a **Cauchy sequence**

[1] I assume V is a real vector space in what follows, there are suitable modifications of these to complex and other contexts, but our focus on the real case here

if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ for which $m, n > N$ implies $||x_m - x_n|| < \epsilon$. Generally, any convergence sequence will be a Cauchy sequence. However, to obtain the converse claim that Cauchy sequences are convergent is only true for **complete** spaces. Indeed, this is a tautological claim as the definition of **complete** is that each Cauchy sequence converges. A complete normed linear space is called a **Banach space**. In advanced calculus, I'll show how we can write a theory of differential calculus on a finite-dimensional Banach space. The abstract study of metric spaces is usually seen in real and functional analysis.

This section has focused on different structures which we can study on the point-set $\mathbb{R}^3$. In the remainder of this chapter we mostly focus on the tangent space, or tangent spaces attached along some object. Tangent space is a vector space and will make great use of the inner-product space structure described in the section which follows.

2.2 vectors and frames in three dimensions

In this section we exploit the isomorphism from $\mathbb{R}^3$ to $T_p\mathbb{R}^3$ to lift all our favorite vector constructions to the tangent space; essentially the point p is either ignored, or just rides along:

Definition 2.2.1. *dot and cross product on $T_p\mathbb{R}^3$*

Let $p \in \mathbb{R}^3$ and suppose $(p, v), (p, w) \in T_p\mathbb{R}^3$ then we define

$$(p, v) \bullet (p, w) = v \bullet w \qquad \& \qquad (p, v) \times (p, w) = (p, v \times w)$$

Define the **Levi-Civita** by $\epsilon_{123} = 1$ and all other ϵ_{ijk} are obtained by assuming that ϵ_{ijk} is completely antisymmetric then we find any repeat of indices causes $\epsilon_{ijk} = 0$ and the nontrivial terms are given by:

$$1 = \epsilon_{123} = \epsilon_{231} = \epsilon_{312} \qquad \& \qquad -1 = \epsilon_{321} = \epsilon_{213} = \epsilon_{132}$$

Hence the Levi-Civita symbol allows an elegant formula for the cross product:

$$v \times w = \sum_{ijk} \epsilon_{ijk} v^i w^j \partial_k|_p \quad \text{or} \quad (v \times w)^k = \sum_{ij} \epsilon_{ijk} v^i w^j$$

where we assume $v, w \in T_p\mathbb{R}^3$ are expressed as $v = \sum_{i=1}^3 v^i \frac{\partial}{\partial x^i}\big|_p$ and $w = \sum_{j=1}^3 w^j \frac{\partial}{\partial x^j}\big|_p$. We also have a concise formula for the dot-product: once more, for $v, w \in T_p\mathbb{R}^3$ as above

$$v \bullet w = \sum_{i=1}^{3} v^i w^i.$$

The identity below is shown in O'neill without resorting to *tricks*. Let me be tricky. First, I invite the reader to observe $\sum_k \epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$. With this settled, calculate:

$$||v \times w||^2 = \sum_k (v \times w)^k (v \times w)^k = \sum_{ijklm} \epsilon_{ijk}\epsilon_{klm} v^i w^j v^l w^m = \sum_{ijlm} (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}) v^i w^j v^l w^m$$

But, $\sum_{ijlm} \delta_{il}\delta_{jm} v^i w^j v^l w^m = \sum_i v^i v^i \sum_j w^j w^j = (v \bullet v)(w \bullet w)$. Likewise, we calculate that $\sum_{ijlm} \delta_{jl}\delta_{im} v^i w^j v^l w^m = \sum_i v^i w^i \sum_j w^j v^j = (v \bullet w)^2$. Hence,

$$||v \times w||^2 = (v \bullet v)(w \bullet w) - (v \bullet w)^2$$

Of course, the same identity is exists for $T_p\mathbb{R}^3$.

In multivariate calculus we defined the length of v to be $||v|| = \sqrt{v \bullet v}$ hence:

Definition 2.2.2. *vector norm in $T_p\mathbb{R}^3$*

Let $p \in \mathbb{R}^3$ and suppose $(p, v) \in T_p\mathbb{R}^3$ where $v = \sum_{i=1}^3 v^i \frac{\partial}{\partial x^i}\big|_p$. We define

$$||(p, v)|| = ||v|| = \sqrt{v \bullet v} = \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2}.$$

The length of a vector is also known as the **norm**. Let me review what we know about dot and cross products in $\mathbb{R}^3$. We have both triangle and Cauchy-Schwarz inequalities:

$$||v + w|| \leq ||v|| + ||w|| \qquad \& \qquad |v \bullet w| \leq ||v||\,||w||.$$

The Cauchy-Schwarz inequality allows us to define the angle between non-zero vectors as

$$\left| \frac{v \bullet w}{||v||\,||w||} \right| \leq 1$$

implies the quotient may be identified with $\cos\theta$ for $0 \leq \theta \leq \pi$. Geometrically, $\cos\theta = \frac{v \bullet w}{||v||\,||w||}$ may also be seen from the law of cosines. Next, recall $||v \times w||^2 = (v \bullet v)(w \bullet w) - (v \bullet w)^2$ hence $||v \times w|| = ||v||^2\,||w||^2 \sin^2\theta$ as $1 - \cos^2\theta = \sin^2\theta$. You should recall the direction of $v \times w$ was given by the **right-hand-rule**. Now, since $T_p\mathbb{R}^3$ also has the dot and cross products, we know:

$$(p, v) \bullet (p, w) = ||(p, v)||\,||(p, w)|| \cos\theta \qquad \& \qquad ||(p, v) \times (p, w)|| = ||(p, v)||\,||(p, w)||\,|\sin\theta|.$$

In truth, we use $T_p\mathbb{R}^3$ in university physics. It is common place for us to take dot and cross products of vectors attached to points away from the origin.

Definition 2.2.3. *orthogonal vectors $T_p\mathbb{R}^3$*

If $(p, v), (p, w) \in T_p\mathbb{R}^3$ and $(p, v) \bullet (p, w) = 0$ then we say (p, v) and (p, w) are **orthogonal**. If $S = \{(p, v_i) \mid i = 1, \dots, k\}$ then we say S is an **orthogonal** set of vectors if $(p, v_i) \bullet (p, v_j) = 0$ for all $i \neq j$. If S is orthogonal and $||(p, v_i)|| = 1$ for each $(p, v_i) \in S$ then we say S is an **orthonormal** set of vectors in $T_p\mathbb{R}^3$.

Notice $(p, 0)$ is orthogonal to every vector in $T_p\mathbb{R}^3$. If we have a set of three orthonormal vectors at $p \in \mathbb{R}^3$ then we have a very nice basis for $T_p\mathbb{R}^3$. Recall **basis** means the set is a linearly independent spanning set for the vector space in question. Certainly orthonormality of $\{v_1, v_2, v_3\} \subseteq T_p\mathbb{R}^3$ implies linear independence as:

$$c_1v_1 + c_2v_2 + c_3v_3 = 0 \quad \Rightarrow \quad c_1v_1 \bullet v_j + c_2v_2 \bullet v_j + c_3v_3 \bullet v_j = 0 \bullet v_j = 0$$

and orthonormality gives $v_i \bullet v_j = \delta_{ij}$ hence only the j-th term remains to give $c_j = 0$. But, j was arbitrary hence $\{v_1, v_2, v_3\}$ is linearly independent. Moreover, as $T_p\mathbb{R}^3$ is isomorphic to the three dimensional vector space $\mathbb{R}^3$ we find that $\{v_1, v_2, v_3\}$ is also a spanning set. Then as $\{v_1, v_2, v_3\}$ is a spanning set for each $X \in T_p\mathbb{R}^3$ there exist $c^1, c^2, c^3 \in \mathbb{R}$ for which $X = c^1v_1 + c^2v_2 + c^3v_3$. But, taking the dot-product with v_j shows $c^j = X \bullet v_j$ for $j = 1, 2, 3$ hence we find:

$$X = (X \bullet v_1)v_1 + (X \bullet v_2)v_2 + (X \bullet v_3)v_3 \tag{2.1}$$

In summary, a set of three orthonormal vectors for $T_p\mathbb{R}^3$ forms an **orthonormal basis**. Such a basis is very convenient as it allows calculation of dot-products in the same fashion as the standard $\{\partial/\partial x^1|_p, \partial/\partial x^2|_p, \partial/\partial x^3|_p\}$ basis. Let us make use of O'neill's notation U_1, U_2, U_3 in place of $\partial/\partial x^1|_p, \partial/\partial x^2|_p, \partial/\partial x^3|_p$. Thus, given $X, Y \in T_p\mathbb{R}^3$, we can either expand in the standard basis

$$X = X^1U_1 + X^2U_2 + X^3U_3 \qquad \& \qquad Y = Y^1U_1 + Y^2U_2 + Y^3U_3$$

or with respect to the orthonormal basis[2] $\{E_1, E_2, E_3\}$

$$X = a^1E_1 + a^2E_2 + a^3E_3 \qquad \& \qquad Y = b^1E_1 + b^2E_2 + b^3E_3$$

then the dot-product of X and Y in the standard coordinates is $X \bullet Y = X^1Y^1 + X^2Y^2 + X^3Y^3$. Likewise, as $E_i \bullet E_j = \delta_{ij}$ we derive

$$\begin{aligned} X \bullet Y &= \sum_i a^iE_i \bullet \sum_j b^jE_j \\ &= \sum_{i,j} a^ib^jE_i \bullet E_j \\ &= \sum_{i,j} a^ib^j\delta_{ij} \\ &= \sum_i a^ib^i = a^1b^1 + a^2b^2 + a^3b^3. \end{aligned}$$

Let us record our result for future reference:

Proposition 2.2.4. *dot-product with respect to orthonormal basis.*

If E_1, E_2, E_3 is a frame and $X = a^1E_1 + a^2E_2 + a^3E_3$ and $Y = b^1E_1 + b^2E_2 + b^3E_3$ then

$$X \bullet Y = a^1b^1 + a^2b^2 + a^3b^3.$$

Suppose $E_1 \times E_2 = E_3$. Since $E_2 \times E_3 \in T_p\mathbb{R}^3$ we may expand it in the orthonormal basis $\{E_1, E_2, E_3\}$,

$$E_2 \times E_3 = [E_1 \bullet (E_2 \times E_3)]E_1 + [E_2 \bullet (E_2 \times E_3)]E_2 + [E_3 \bullet (E_2 \times E_3)]E_3$$

Next, we use $A \bullet (B \times C) = B \bullet (C \times A) = C \bullet (A \times B)$ to simplify what follows:

$$\begin{aligned} E_2 \times E_3 &= [E_3 \bullet (E_1 \times E_2)]E_1 + [E_3 \bullet (E_2 \times E_2)]E_2 + [E_2 \bullet (E_3 \times E_3)]E_3 \\ &= [E_3 \bullet E_3]E_1 \\ &= E_1. \end{aligned}$$

A similar calculation shows $E_3 \times E_1 = E_2$. Indeed, there is nothing special about assuming $E_1 \times E_2 = E_3$ as our starting point. Any one of the following necessitates the remaining pair

$$E_1 \times E_2 = E_3 \qquad \& \qquad E_2 \times E_3 = E_1 \qquad \& \qquad E_3 \times E_1 = E_2.$$

provided we know E_1, E_2, E_3 are orthonormal. Concisely, we have $E_i \times E_j = \sum_k \epsilon_{ijk}E_k$. Such a triple of vectors is sometimes called a **right-handed-triple** in physics. If $X = \sum_i a^iE_i$ and

[2] you could use any notation you like here, I'll try to stick with capital E or F for these as to follow O'neill.

$Y = \sum_j b^j E_j$ with respect to a right-handed orthonormal basis then the formula for the $X \times Y$ is the same as for the Cartesian coordinate system:

$$X \times Y = \sum_i a^i E_i \times \sum_j b^j E_j = \sum_{i,j} a^i b^j E_i \times E_j = \sum_{i,j,k} \epsilon_{ijk} a^i b^j E_k.$$

Where E_i is in place of U_i and the components $a^i = X \bullet E_i$ and $b^j = Y \bullet e_j$. If the basis is orthonormal, but, not right-handed then it turns out that it must be the case that $E_1 \times E_2 = -E_3$ and the formula above picks up a minus. Once again, let us record our result for future reference:

Proposition 2.2.5. *cross-product with respect to orthonormal basis.*

If E_1, E_2, E_3 is a frame with $E_1 \times E_2 = E_3$ and $X = a^1 E_1 + a^2 E_2 + a^3 E_3$ and $Y = b^1 E_1 + b^2 E_2 + b^3 E_3$ then $X \times Y = (a^2 b^3 - a^3 b^2) E_1 + (a^3 b^1 - a^1 b^3) E_2 + (a^1 b^2 - a^2 b^1) E_3$.. If $E_1 \times E_2 = -E_3$ then $X \times Y = -(a^2 b^3 - a^3 b^2) E_1 - (a^3 b^1 - a^1 b^3) E_2 - (a^1 b^2 - a^2 b^1) E_3$.

In differential geometry, the assignment of such a triple of vectors is known as *attaching a frame* to p. Actually, in the larger scheme of things I would tend to call it an *orthonormal frame*. But, since all the frames we work with in this course are orthonormal, the omission of "orthonormal" seems fair. We will be interested in framing curves and surfaces at each point. A single frame contains 3 vector fields, so, you might wonder why we bother introducing yet another object. The reason will be clear soon enough; frames are naturally connected to very special matrices. First, a definition:

Definition 2.2.6. *frames in* $\mathbb{R}^3$

If $\{E_1, E_2, E_3\}$ is an orthonormal basis for $T_p\mathbb{R}^3$ then we say $\{E_1, E_2, E_3\}$ is a **frame at** p. A **frame on** S is an single-valued assignment of a frame $\{E_1(p), E_2(p), E_3(p)\}$ to each $p \in S \subseteq \mathbb{R}^3$. A **positively oriented frame** has $E_1 \times E_2 = E_3$.

Naturally, the coordinate vector fields provide a frame.

Example 2.2.7. *Let $p \in \mathbb{R}^3$ then E_1, E_2, E_3 given below form a frame at p*

$$E_1 = \frac{1}{\sqrt{3}}\left(\frac{\partial}{\partial x}\bigg|_p + \frac{\partial}{\partial y}\bigg|_p + \frac{\partial}{\partial z}\bigg|_p\right), \quad E_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x}\bigg|_p - \frac{\partial}{\partial z}\bigg|_p\right), \quad E_3 = \frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial x}\bigg|_p - 2\frac{\partial}{\partial y}\bigg|_p + \frac{\partial}{\partial z}\bigg|_p\right)$$

Example 2.2.8. *Observe $\{\partial_x, \partial_y, \partial_z\}$ forms the* **Cartesian coordinate frame** *on $\mathbb{R}^3$. We sometimes denote this frame by the standard notation $\{U_1, U_2, U_3\}$. It is often useful to express a given frame in terms of the Euclidean frame. For example, the frame of the preceding example is written as:*

$$E_1 = \tfrac{1}{\sqrt{3}}\left(U_1 + U_2 + U_3\right), \quad E_2 = \tfrac{1}{\sqrt{2}}\left(U_1 - U_3\right), \quad E_3 = \tfrac{1}{\sqrt{6}}\left(U_1 - 2U_2 + U_3\right).$$

Example 2.2.9. *The* **cylindrical coordinate frame** *is given below:*

$$\begin{aligned} E_1 &= \cos\theta\, U_1 + \sin\theta\, U_2 \\ E_2 &= -\sin\theta\, U_1 + \cos\theta\, U_2 \\ E_3 &= U_3. \end{aligned}$$

I often use the notation $E_1 = \widehat{r}, E_2 = \widehat{\theta}$ and $E_3 = \widehat{z}$ in multivariate calculus. This frame is very useful for simplifying calculations with cylindrical symmetry.

Example 2.2.10. *The* **spherical coordinate frame** *for the usual spherical coordinates used in third-semester-American calculus is given below:*

$$\begin{aligned} E_1 &= \cos\theta\sin\phi\, U_1 + \sin\theta\sin\phi\, U_2 + \cos\phi\, U_3 \\ E_2 &= \cos\theta\cos\phi\, U_1 + \sin\theta\cos\phi\, U_2 - \sin\phi\, U_3 \\ E_3 &= -\sin\theta\, U_1 + \cos\theta\, U_2 \end{aligned}$$

I often use the notation $E_1 = \widehat{\rho}, E_2 = \widehat{\phi}$ *and* $E_3 = \widehat{\theta}$ *in multivariate calculus. This frame is very useful for simplifying calculations with spherical symmetry.*

I should warn the readers of O'neill, he uses a different choice of spherical coordinates than we implicitly use in the example above. In fact, the example is based on the formulas:

$$x = \rho\cos\theta\sin\phi, \qquad y = \rho\sin\theta\sin\phi, \qquad z = \rho\cos\phi$$

for $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. These coordinates envision ϕ being zero on the positive z-axis then sweeping down to π on the negative z-axis. In contrast, see Figure 2.20 on page 86, O'neill prefers to work with ϕ which is zero on the xy-plane then sweeps up or down to $\pm\pi/2$.

The attitude matrix places Cartesian coordinate vectors of a frame as rows of the matrix. My natural inclination would be to use the transpose of this, but, what follows is a standard construction.

Definition 2.2.11. *attitude of a frame*

If $\{E_1, E_2, E_3\}$ is a frame of $T_p\mathbb{R}^3$ and

$$E_i = a_{i1}U_1 + a_{i2}U_2 + a_{i3}U_3$$

for $i = 1, 2, 3$. Then define the **attitude matrix** of the frame by:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Notice, $E_i = a_{i1}U_1 + a_{i2}U_2 + a_{i3}U_3$ implies $a_{ij} = E_i \bullet U_j$. Therefore, the definition above can be formulated as $A_{ij} = E_i \bullet U_j$ for $1 \leq i, j \leq 3$. Naturally, we may either consider the attitude matrix at a point, or, if we wish to allow p to vary then A contains nine functions which describe how the frame varies with p. We may at times replace A with $A(p)$ to emphasize the point-dependence.

We need to recall the transpose of a matrix is defined by $(A^T)_{ij} = a_{ji}$ and an **orthogonal matrix** is M such that $M^TM = I$ where I is the identity matrix.

Theorem 2.2.12. *attitude matrix is an orthogonal matrix .*

If $A = (a_{ij})$ is an attitude matrix then $\displaystyle\sum_{k=1}^{3} a_{ki}a_{kj} = \delta_{ij}$. That is, $A^TA = I$.

Proof: let A be an attitude matrix where $A_{ij} = E_i \bullet U_j$. Consider,

$$\begin{aligned}(A^T A)_{ij} &= \sum_k (A^T)_{ik} A_{kj} \\ &= \sum_k a_{ki} a_{kj} \\ &= \sum_k (U_k \bullet E_i)(U_k \bullet E_j) \\ &= E_i \bullet E_j \\ &= \delta_{ij}.\end{aligned}$$

Going from the third to fourth line we have identified $E_i = \sum_k (U_k \bullet E_i) U_k$ is the expansion of E_i in the Cartesian frame hence the next step is the definition of the dot-product. $\square$

Now we return to our frame examples to extract the attitude matrix. In each case, I invite the reader to verify $A^T A = I$.

Example 2.2.13. *Following Example 2.2.7,*

$$\begin{array}{l} E_1 = \frac{1}{\sqrt{3}}\left(U_1 + U_2 + U_3\right), \\ E_2 = \frac{1}{\sqrt{2}}\left(U_1 - U_3\right), \\ E_3 = \frac{1}{\sqrt{6}}\left(U_1 - 2U_2 + U_3\right) \end{array} \quad \Rightarrow \quad A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Example 2.2.14. *Following Ex. 2.2.8, the attitude of the Cartesian frame is the identity matrix:*

$$\begin{array}{l} E_1 = 1 \cdot U_1 + 0 \cdot U_2 + 0 \cdot U_3, \\ E_2 = 0 \cdot U_1 + 1 \cdot U_2 + 0 \cdot U_3, \\ E_3 = 0 \cdot U_1 + 0 \cdot U_2 + 1 \cdot U_3 \end{array} \quad \Rightarrow \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 2.2.15. *Following Ex. 2.2.9, the attitude of the* **cylindrical coordinate frame** *is:*

$$\begin{array}{l} E_1 = \cos\theta\, U_1 + \sin\theta\, U_2 \\ E_2 = -\sin\theta\, U_1 + \cos\theta\, U_2 \\ E_3 = U_3. \end{array} \quad \Rightarrow \quad A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 2.2.16. *Following Ex. 2.2.10, the attitude of the* **spherical coordinate frame** *is:*

$$\begin{array}{l} E_1 = \cos\theta\sin\phi\, U_1 + \sin\theta\sin\phi\, U_2 + \cos\phi\, U_3 \\ E_2 = \cos\theta\cos\phi\, U_1 + \sin\theta\cos\phi\, U_2 - \sin\phi\, U_3 \\ E_3 = -\sin\theta\, U_1 + \cos\theta\, U_2 \end{array} \quad \Rightarrow \quad A = \begin{bmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix}.$$

From the first pair of examples we see the attitude matrix of a constant frame is likewise constant. From the latter pair the attitude is variable when the frame is variable. We have much more to say about this structure in the remainder of this chapter. The next section is about how to differentiate along a curve, but, then the section after it is all about *framing a curve.*

2.3 calculus of vectors fields along curves

To begin, we introduce some standard terminology for vector fields along a curve.

Definition 2.3.1. *vector field and Cartesian components:*

Let $\alpha : I \to \mathbb{R}^3$ by a smooth parametrized curve and suppose $Y = \sum_i Y^i \partial_i$ is a smooth vector field on the curve then we say $Y \in \mathfrak{X}(\alpha)$. That is, $\mathfrak{X}(\alpha)$ is the set of all smooth vector fields on α. The functions $Y^i : U \subseteq \mathbb{R}^3 \to \mathbb{R}$ are the **Cartesian coordinate functions** of Y. The functions $Y^i \circ \alpha : I \to \mathbb{R}$ are the **parameterized components of Y along α**.

To say Y is *smooth* is to say it has smooth coordinate functions[3].

Example 2.3.2. *Let $\alpha(t) = (t, t^2, t^3)$ for $t \in \mathbb{R}$ and $Y = x^2 \partial_x + (y + \sin(z))\partial_z$ then identify we have vector field component functions:*

$$Y^1 = x^2, \qquad Y^2 = 0, \qquad Y^3 = y + \sin(z)$$

which give parametrized components on $\alpha(t) = (t, t^2, t^3)$ of

$$(Y^1 \circ \alpha)(t) = t^2, \qquad (Y^2 \circ \alpha)(t) = 0, \qquad (Y^3 \circ \alpha)(t) = t^2 + \sin(t^3).$$

To differentiate Y along the parametrized curve α we simply differentiate its parametrized components: as usual, we could use U_i or ∂_i in what follows:

Definition 2.3.3. *change in a vector field along a parameterized curve:*

Let $\alpha : I \to \mathbb{R}^3$ by a smooth parametrized curve and suppose $Y = \sum_i Y^i \partial_i$ is a vector field defined on some open set containing the curve. We define the **derivative of Y along α** as follows:

$$Y'(t) = \sum_i \frac{d(Y^i \circ \alpha)}{dt} \frac{\partial}{\partial x^i}\bigg|_{\alpha(t)} \in T_{\alpha(t)}\mathbb{R}^3.$$

If we have $Y(\alpha(t)) = a(t)\,U_1 + b(t)\,U_2 + c(t)\,U_3$ then

$$\frac{dY}{dt} = \frac{da}{dt}U_1 + \frac{db}{dt}U_2 + \frac{dc}{dt}U_3$$

where to be clear, the Cartesian frame is at the point $\alpha(t)$.

Example 2.3.4. *Continuing Example 2.3.2, the vector field along α is given by*

$$(Y \circ \alpha)(t) = t^2 U_1 + \left(t^2 + \sin(t^3)\right) U_3$$

thus $Y'(t) = 2t\,U_1 + (2t + 3t^2\cos(t^3))U_3 \in T_{(t,t^2,t^3)}\mathbb{R}^3$.

[3]this definition, like most we encounter, is improved in deeper study of manifolds

Proposition 2.3.5. *calculus of vector fields on curves.*

Let α is a smooth parametrized curve and $Y, Z \in \mathfrak{X}(\alpha)$ and $c_1, c_2 \in \mathbb{R}$,

(i.) $\dfrac{d}{dt}(c_1 Y + c_2 Z) = c_1 \dfrac{dY}{dt} + c_2 \dfrac{dZ}{dt}$,

(ii.) $\dfrac{d}{dt}(Y \bullet Z)(\alpha(t)) = \dfrac{dY}{dt} \bullet Z(\alpha(t)) + Y(\alpha(t)) \bullet \dfrac{dZ}{dt}$,

(iii.) $\dfrac{d}{dt}(Y \times Z) = \dfrac{dY}{dt} \times Z(\alpha(t)) + Y(\alpha(t)) \times \dfrac{dZ}{dt}$.

Proof: if $Y \circ \alpha = \sum_i a^i U_i$ and $Z \circ \alpha = \sum_i b^i U_i$ then $(Y \bullet Z) \circ \alpha = \sum_i a^i b^i$ thus calculate:

$$\begin{aligned}
\frac{d}{dt}(Y \bullet Z) \circ \alpha &= \frac{d}{dt} \sum_i a^i b^i \\
&= \sum_i \frac{d}{dt}(a^i b^i) \\
&= \sum_i \left[\frac{da^i}{dt} b^i + a^i \frac{db^i}{dt} \right] \\
&= \sum_i \frac{da^i}{dt} b^i + \sum_i a^i \frac{db^i}{dt} \\
&= \frac{dY}{dt} \bullet Z(\alpha(t)) + Y(\alpha(t)) \bullet \frac{dZ}{dt}.
\end{aligned}$$

Since $(Y \times Z) \circ \alpha = \sum_{i,j,k} \epsilon_{ijk} a^i b^j U_k$ we can prove (3.) in a similar fashion. I leave (1.) to the reader. $\square$

In Chapter 1 we studied how a given parametrized curve naturally generates an associated vector field which is known as the **velocity** of the curve. If we differentiate the velocity field along α this generates the **acceleration** which is also a vector field along α.

Definition 2.3.6. *acceleration of parametrized curve:*

Let $\alpha : I \to \mathbb{R}^3$ by a smooth parametrized curve then define $\alpha''(t) = \frac{d}{dt}(\alpha'(t))$. That is:

$$\alpha''(t) = \sum_i \frac{d^2(Y^i \circ \alpha)}{dt^2} \frac{\partial}{\partial x^i}\bigg|_{\alpha(t)} \in T_{\alpha(t)}\mathbb{R}^3.$$

The definitions of acceleration and velocity we give in this section should be recognizable from multivariate calculus, or university physics. However, the concept of differentiating a vector field along the curve and fitting the acceleration in the context of that construction is probably new.

Example 2.3.7. *Let $\alpha(t) = (t, t^2, t^3)$ for $t \in \mathbb{R}$. Then*

$$\alpha'(t) = U_1 + 2t\, U_2 + 3t^2\, U_3 \qquad \& \qquad \alpha''(t) = 2\, U_2 + 6t\, U_3.$$

where both $\alpha'(t)$ and $\alpha''(t)$ are in $T_{\alpha(t)}\mathbb{R}^3$.

The concept of **distant parallelism** is fairly easy to grasp in $\mathbb{R}^n$. In particular, we say (p, v) and (q, w) are **distantly parallel** if $p \neq q$ and $v \bullet w = ||v||\,||w||$. That is, for $p \neq q$, the vectors (p, v) and (q, w) are distantly parallel if there vector parts are parallel. For example, given a non-intersecting parametrized curve α, if a vector field Y has $Y(\alpha(t)) = \left(c^1\partial_x + c^2\partial_y + c^3\partial_z\right)|_{\alpha(t)}$ for constants c^1, c^2, c^3 then $Y(\alpha(t_1))$ and $Y(\alpha(t_2))$ are distantly parallel.

Theorem 2.3.8.

Let $\alpha : I \to \mathbb{R}^3$ is a smooth parametrized curve,

(i.) $\alpha' = 0$ iff $\alpha(t) = p$ for all $t \in I$ (that is, α is a point),

(ii.) $\alpha'' = 0$ iff $\alpha(t) = p + tv$ for all $t \in I$ (that is, α is a line),

(iii.) Let $Y \in \mathfrak{X}(\alpha)$ then the derivative of Y along α is zero iff $Y \circ \alpha = c^1U_1 + c^2U_2 + c^3U_3$ for constants c^1, c^2, c^3.

Proof of (1.): if $\alpha' = 0$ then for $i = 1, 2, 3$ we have $\frac{d\alpha^i}{dt} = 0$ for all $t \in I$. We assume I is connected hence $\alpha^i(t) = p^i$ for all $t \in I$ thus $\alpha(t) = p$ for all $t \in I$. Conversely $\alpha(t) = p$ clearly implies $\alpha' = 0$.

Proof of (2.): almost the same as (1.), just have to integrate twice in the forward direction and clearly $\alpha(t) = p + tv$ for $p, v \in \mathbb{R}^3$ has $\alpha''(t) = 0$. I leave the details to the reader.

Proof of (3.): let $Y \in \mathfrak{X}(\alpha)$ have constant length along α. Suppose the derivative of Y along α is zero. That is, suppose $\frac{d}{dt}(Y \circ \alpha) = 0$. Let $(Y \circ \alpha)^i = a^i$ then we are given $\frac{da^i}{dt} = 0$ for $t \in I$. Observe, for $i = 1, 2, 3$, $\frac{da^i}{dt} = 0$ for all $t \in I$ implies $a^i = c^i$ for some constant c^i. Therefore, $Y \circ \alpha = c^1U_1 + c^2U_2 + c^3U_3$. Conversely, it is simple to see that $Y \circ \alpha = c^1U_1 + c^2U_2 + c^3U_3$ for constants c^1, c^2, c^3 implies $\frac{d}{dt}(Y \circ \alpha)(t) = 0$ hence $Y' = 0$. $\square$

Given that $Y \in \mathfrak{X}(\alpha)$ has constant length along α we do have the equivalence: $Y' = 0$ iff $Y(\alpha(t_1))$ is distantly parallel to $Y(\alpha(t_2))$. The addition of the assumption of constant length is needed since it is possible to have vector fields which are parallel along α but have variable length. For example, take $\alpha(t) = (t, t, t)$ for $t \in [1, 2]$ and consider $Y = xU_1$. We have $Y(\alpha(t)) = tU_1$ thus, for any $t \in [1, 2]$ the vector $Y(\alpha(t))$ is distantly parallel[4] to U_1. However, $Y' = U_1 \neq 0$. In short, I argue part (3.) of Lemma 2.3 on page 56 of O'neill (where no mention of Y's length is made) is false unless we insist Y have constant length.

2.4 Frenet Serret frame of a curve

We have the necessary toys. Finally, let us play. We wish to find a frame along a curve. Lines are too boring, there is no nice way to pick a direction in the normal plane at a given point. For $\alpha(t) = p + tv$ it is easy enough to see $\alpha'(t) = (p + tv, v)$, but, beyond the direction vector v, how to we find another characteristic vector for the line? This problem is either too easy or too hard for us, so, we set it aside and **assume in the remainder of this section we have a non-linear, regular, parametrized curve parametrized by arclength**. That is, we have a non-linear, unit-speed curve. We wish to find a frame along α.

[4] you might notice distant parallelism is an equivalence relation

Our goal is to find a frame E_1, E_2, E_3 along α; that is $E_1, E_2, E_3 \in \mathfrak{X}(\alpha)$ for which E_1, E_2, E_3 forms an orthonormal basis for $T_{\alpha(t)}\mathbb{R}^3$ at each $t \in \text{dom}(\alpha)$. We'll start with the velocity of the curve. Let us denote the **unit-tangent** field by $T = \alpha'(s)$ then $T \bullet T = ||\alpha'(s)|| = 1$. Next, we need to find another vector field along α. Notice $T' \bullet T + T \bullet T' = 0$ on $\alpha(t)$ hence T' is orthogonal to T along the curve. However, $||T'||$ is not generally 1 thus we must normalize and define the **normal** vector field by $N = \frac{1}{||T'||}T'$. In fact, geometrically, the change in the tangent vector along the curve measures how the curve is changing direction. We define:

Definition 2.4.1. *curvature of smooth curve.*

Suppose α is an arclength-parametrized curve then the **curvature of** α is given by $\kappa = ||T'||$.

Notice, if $\alpha'' = 0$ then $T' = \alpha'' = 0$ hence $\kappa = 0$ for a line. We are primarily interested in the case $\kappa > 0$. Observe, by construction, $N \bullet N = 1$ and $N \bullet T = 0$. Finally, we define the **binormal** $B = T \times N$. Clearly, $B \bullet T = 0$ and $B \bullet N = 0$ hence $||B|| = ||T||||N|| \sin(90^o) = 1$. Thus T, N, B forms a positively oriented frame along α known as the **Frenet frame** of α. The term *positively oriented* refers to the fact that the triple T, N, B respects the right-hand-rule as $B = T \times N$. The tangent and normal vector fields to the curve both are tangent to the so-called *osculating plane*. This makes the binormal a vector which is perpendicular to the osculating plane. That is, the binormal forms the normal to the osculating plane. We'll see in the proof of the proposition to follow that there is a function τ which governs the evolution of B with the curve. That function is called the **torsion**. In short, it measures the tendency of the curve to bend off its osculating plane.

Theorem 2.4.2. *Frenet Serret Equations.*

Let α be a unit-speed, non-linear curve and define $T = \frac{d\alpha}{ds}$ and $N = \frac{1}{||T'||}\frac{dT}{ds}$ and $B = T \times N$ then there exists a function τ for which:

$$\frac{dT}{ds} = \kappa\, N$$
$$\frac{dN}{ds} = -\kappa\, T + \tau\, B$$
$$\frac{dB}{ds} = -\tau N$$

Proof: begin by noting $\kappa = ||T'||$ hence $N = \frac{1}{\kappa}\frac{dT}{ds}$ hence $\frac{dT}{ds} = \kappa N$ is just the definition of κ. The other two equations require a bit more effort.

We wish to show T, N, B forms a frame along α. To verify this, observe $T \bullet T = ||\alpha'(s)|| = 1$ as α is given to be unit-speed. Differentiate by part (ii.) of Proposition 2.3.5,

$$T \bullet T = 1 \quad \Rightarrow \quad T' \bullet T + T \bullet T' = 2T \bullet T' = 0 \quad \Rightarrow \quad T \bullet \left[\frac{T'}{||T'||}\right] = T \bullet N = 0.$$

By construction, $N \bullet N = \frac{1}{||T'||^2}T' \bullet T' = 1$. By properties of the cross-product, B is orthogonal to both N and T as $B = T \times N$. Finally, $||B|| = ||T \times N|| = ||T||||N|| \sin(90^o) = 1$ hence $B \bullet B = 1$. Sorry to be redundant here, but, to be safe I want all of this here.

Notice, N' and B' are vector fields along α by their construction. It follows we can expand each of them in terms of the frame T, N, B along α. Moreover, the components with respect to this orthonormal basis are simply given by dot-products. Begin with B' we have:

$$\boxed{B' = (B' \bullet T)T + (B' \bullet N)N + (B' \bullet B)B} \qquad \star_B$$

notice $B \bullet B = 1$ implies $B' \bullet B = 0$. Also, $B \bullet T = 0$ implies

$$B' \bullet T + B \bullet T' = 0 \quad \Rightarrow \quad B' \bullet T = -B \bullet T' = -B \bullet (\kappa N) = -\kappa B \bullet N = 0.$$

We deduce from $\star_B$ that $B' = (B' \bullet N)N$. Let $\tau = -B' \bullet N$ and we obtain $\frac{dB}{ds} = -\tau N$.

Once more use that T, N, B forms a frame on α and $N' \in \mathfrak{X}(\alpha)$,

$$\boxed{N' = (N' \bullet T)T + (N' \bullet N)N + (N' \bullet B)B} \qquad \star_N$$

As usual, $N \bullet N = 1$ implies $N' \bullet N = 0$. On the other hand, $N \bullet T = 0$ yields,

$$N' \bullet T + N \bullet T' = 0 \quad \Rightarrow \quad N' \bullet T = -T' \bullet N = -\kappa N \bullet N = -\kappa.$$

Whereas, $N \bullet B = 0$ differentiates to yield:

$$N' \bullet B + N \bullet B' = 0 \quad \Rightarrow \quad N' \bullet B = -N \bullet B' = -N \bullet (-\tau N) = \tau.$$

Therefore, returing to $\star_N$ we find $\frac{dN}{dS} = -\kappa T + \tau B$. $\square$

This proof is really not that complicated. The boxed equations are immediate once we recognize T, N, B forms a frame. Then, we just used part (ii.) of Proposition 2.3.5 repeatedly to cut the components of the boxed equation down to size. We have shown the definition of torsion to follow is well-posed:

Definition 2.4.3. *torsion of smooth curve.*

Let α is an arclength-parametrized curve then the **torsion of** α is given by $\tau = -N \bullet \frac{dB}{ds}$.

Example 2.4.4. *Consider the helix defined by $R, m > 0$ and*

$$\alpha(s) = (R\cos(ks), R\sin(ks), mks)$$

for $s \in \mathbb{R}$ and $k = 1/\sqrt{R^2 + m^2}$. Calculate,

$$\alpha'(s) = -kR\sin(ks)\, U_1 + kR\cos(ks)\, U_2 + mkU_3$$

thus $||\alpha'(s)|| = k\sqrt{R^2 + m^2} = 1$. It follows $T = \alpha'$. Differentiate α' to obtain:

$$T\,'(s) = \alpha''(s) = -k^2 R\cos(ks)\, U_1 - k^2 R\sin(ks)\, U_2$$

We find $||T'(s)|| = k^2 R$ hence $\kappa = R/(R^2 + m^2)$. Note $N = -\cos(ks)\, U_1 - \sin(ks)\, U_2$ thus

$$\begin{aligned} B = T \times N &= \big(-kR\sin(ks)\, U_1 + kR\cos(ks)\, U_2 + mkU_3\big) \times \big(-\cos(ks)\, U_1 - \sin(ks)\, U_2\big) \\ &= mk\sin(ks)\, U_1 - mk\cos(ks)\, U_2 + kR\, U_3. \end{aligned}$$

As a quick check on the calculation, notice $B \bullet N = 0$ and $B \bullet T = 0$. Calculate,

$$\begin{aligned} \frac{dB}{ds} &= \frac{d}{ds}\left[mk\sin(ks)\, U_1 - mk\cos(ks)\, U_2 + kR\, U_3\right] \\ &= mk^2\cos(ks)\, U_1 + mk^2\sin(ks)\, U_2 \end{aligned}$$

Thus $\frac{dB}{ds} \bullet N = -mk^2$ thus $\tau = mk^2 = m/(m^2 + R^2)$.

The nice thing about the helix example is we see nonzero curvature and torsion. Moreover, we can see other features in particular limits of the general formulas. For example, if $m = 0$ then $\alpha(s) = (R\cos(ks), R\sin(ks), 0)$ is just a circle and the torsion $\tau = 0$ as $B = U_3$. A circle is a *planar curve* and the vanishing torsion reflects this fact. Also, when $m = 0$ we have $\kappa = 1/R$. We find the curvature is large when R is small.

Generally, if a curve lies in a plane then surely T and N are tangent to the plane. But, then that gives B is colinear to the normal of the plane which implies B is constant hence $\tau = 0$. Conversely, if $\tau = 0$ for a curve then we have B is constant along the curve which implies T, N are always parallel to a plane with B as a normal. So, it seems plausible that a curve is planar iff it has vanishing torsion. That said, we should be a bit more precise:

Theorem 2.4.5. *no torsion, non-linear curves are planar*

> Let α be a unit-speed, non-linear curve then α is a planar curve iff $\tau = 0$.

Proof ($\Rightarrow$) : if the parametrized curve $\alpha : I \to \mathbb{R}^3$ lies on a plane with normal V then and suppose $s_o \in I$. Then $\alpha(s_o)$ is a point on the plane and we have that $\gamma(s) \bullet V = 0$ for all $s \in I$ where $\gamma(s) = \alpha(s) - \alpha(s_o)$ is the secant line from s_o to s on the given curve. By the product rule and the fact that V is constant we have $\gamma'(s) \bullet V = 0$ hence $T \bullet V = 0$ and differentiating again yields $T' \bullet V = \kappa N \bullet V = 0$ hence $N \bullet V = 0$ as α is non-linear gives $\kappa > 0$. We find V is orthogonal to both T and N for each $s \in I$ hence V is colinear with B at each $s \in I$ and as $B \neq 0$ and $V \neq 0$ there must exist $k \neq 0$ for which $B = kV$. Thus $B' = 0$ and we derive $\tau = -B' \bullet N = 0$.

Proof ($\Leftarrow$) : suppose $\tau(s) = 0$ for all $s \in I$ where $s_o \in I$ and $\alpha : I \to \mathbb{R}^3$ is a non-linear regular curve. Notice $B' = -\tau N = 0$ for all $s \in I$ hence $B(s) = B(s_o)$ for all $s \in I$. We expect that $B(s_o)$ serves as the normal to the plane in which the curve evolves. Thus define a function which if identically zero will show that the curve lies in a plane with point $\alpha(s_o)$ and normal $B(s_o)$. Following O'neill[5],

$$f(s) = [\alpha(s) - \alpha(s_o)] \bullet B(s_o) \quad \Rightarrow \quad \frac{df}{ds} = \alpha'(s) \bullet B(s_o) = T(s) \bullet B(s) = 0$$

for all $s \in I$ however $f(s_o) = 0$ thus $f(s) = 0$ for all $s \in I$ and so we find the curve is planar with the constant binormal serving as the normal to the plane of motion. $\square$

It is instructive to consider how the curvature and torsion locally approximate the geometry of a smooth regular unit-speed curve. Taylor's theorem gives, for some point s_o in the domain of α,

$$\alpha(s) \approx \alpha(s_o) + s\alpha'(s_o) + \frac{s^2}{2}\alpha''(s_o) + \frac{s^3}{6}\alpha'''(s_o) + \cdots \qquad \star$$

Note $\alpha'(s_o) = T(s_o)$ and for our convenience let us denote $T(s_o) = T_o$ hence identify first two terms parametrize the tangent line at $\alpha(s_o)$ by $\alpha(s_o) + sT_o$. Next, by the Frenet Serret equation for $T' = \alpha''$ we see

$$\alpha''(s_o) = T'(s_o) = \kappa(s_o)N(s_o)$$

[5]notice that $\alpha(s) - \alpha(s_o)$ is naturally associated with the directed line-segment in $\mathbb{R}^3$ from $\alpha(s_o)$ to $\alpha(s)$. We attach that directed line-segment to $\alpha(s_o)$ as to make the dot-product with $B(s_o) \in T_{\alpha(s_o)}\mathbb{R}^3$ reasonable. Since little is gained by making these identifications explicit in the proof we follow O'neill and most authors and omit comments such as these in most places (except here I suppose)

let $\kappa(s_o) = \kappa_o$ and $N(s_o) = N_o$. Thus $\star$ reads

$$\alpha(s) \approx \alpha(s_o) + sT_o + \frac{1}{2}s^2\kappa_o N_o + \frac{s^3}{6}\alpha'''(s_o) + \cdots$$

Torsion contributes to the third derivative term. Let's see how. Note, $\alpha'' = T' = \kappa N$ thus

$$\alpha''' = \frac{d\kappa}{ds}N + \kappa N' = \frac{d\kappa}{ds}N + \kappa(-\kappa T + \tau B) = -\kappa^2 T + \frac{d\kappa}{ds}N + \kappa\tau B$$

thus, to third order in arclength,

$$\alpha(s) \approx \alpha(s_o) + \left(s - \tfrac{1}{6}s^3\kappa_o^2\right)T_o + \left(\tfrac{1}{2}s^2\kappa_o + \tfrac{1}{6}s^3\kappa_o'\right)N_o + \tfrac{1}{6}s^3\kappa_o\tau_o B_o + \cdots$$

To just second order in s we have motion in a plane spanned by T_o, N_o. Suppose we let x, y be the T_o, N_o coordinates respective then if $\alpha(s_o) = 0$ we can express

$$x = s \qquad y = \frac{1}{2}s^2\kappa_o \quad \Rightarrow \quad y = \frac{1}{2}\kappa x^2.$$

Thus, the motion follows a parabola with slope κx. We can approximate the motion of the curve at any point in this fashion. Alternatively, we can fit a circle locally to the curve. The circle which fits at $\alpha(s_o)$ is called the **osculating circle**. The osculating circle has radius $1/\kappa_o$. In x, y coordinates as described above, the equation of the osculating circle is:

$$x^2 + \left(y - \frac{1}{\kappa_o}\right)^2 = \frac{1}{\kappa_o^2}.$$

See the background of http://www.supermath.info/MultivariateCalculus.html for a animated picture of an osculating circle attached to a rather twisty space curve. Thanks to my brother Bill for the Maple code. You can read about the *osculating sphere* in Theorem 2.10 on page 22-23 of Wolfgang Kühnel's *Differential Geometry: Curves-Surfaces-Manifolds*.

Suppose the curvature is constant along some segment of a unit-speed regular curve. If the curvature is constantly zero then is a part of a line by part (ii.) of Theorem 2.3.8. On the other hand, if $\kappa(s) = c > 0$ for all s in some interval then the osculating circles along the curve share the same radius and we might hope they are in fact the same circle. As it happens, we also need the curve to be planar, so the assumption $\tau = 0$ is added below:

Theorem 2.4.6. *constant curvature $\kappa \neq 0$ curve follows circle of radius $1/\kappa$*

> Let α be a unit-speed, non-linear curve with constant curvature $\kappa(s) = \kappa_o \neq 0$ and $\tau(s) = 0$ for all $s \in I$ then $\alpha(s)$ follows a circle of radius $1/\kappa_o$.

Proof 1: Let α be a unit-speed, non-linear curve with constant curvature $\kappa(s) = \kappa_o \neq 0$ and $\tau(s) = 0$ for all $s \in I$. Let $s_o \in I$ and denote T_o, N_o to be $T(s), N(s)$ with $s = s_o$. We expect the circle which the curve follows has radius $R = 1/\kappa_o$ thus $C = \alpha(s_o) + RN_o$ should be the center of the circle. Points along the circle should be equidistant from C. As $\tau(s) = 0$ we know $\alpha(s)$ falls on the plane with point $\alpha(s_o)$ and normal $B(s_o)$. It follows there are functions $a(s), b(s)$ for which

$$\alpha(s) = \alpha(s_o) + a(s)T_o + b(s)N_o$$

Thus,

$$\begin{aligned}\alpha(s) - C &= \alpha(s_o) + a(s)T_o + b(s)N_o - [\alpha(s_o) + RN_o] \\ &= a(s)T_o + (b(s) - R)N_o\end{aligned}$$

But, $\alpha'(s) = a'(s)T_o + b'(s)N_o$ and $\alpha''(s) = a''(s)T_o + b''(s)N_o$. Since $\alpha'(s) = T(s)$ and $\alpha''(s) = T'(s) = \kappa(s)N(s) = \kappa_o N(s)$ we have:

$$a''(s)T_o + b''(s)N_o = \kappa_o N(s)$$

differentiating yields

$$a'''(s)T_o + b'''(s)N_o = \kappa_o N'(s) = \kappa_o(-\kappa T + \tau B) = -\kappa_o^2 T(s)$$

however, $-\kappa_o^2 T(s) = -\kappa_o^2[a'(s)T_o + b'(s)N_o]$ thus the equation above yields

$$(a''' + \kappa_o^2 a')T_o + (b''' + \kappa_o^2 b')N_o = 0$$

which means a, b are solutions to the following constant coefficient ODEs,

$$a''' + \kappa_o^2 a' = 0, \qquad b''' + \kappa_o^2 b' = 0$$

Solutions below follow from standard arguments in the introductory DEqns course,

$$a(t) = c_1 + c_2\cos(\kappa_o s) + c_3\sin(\kappa_o s), \qquad b(t) = c_4 + c_5\cos(\kappa_o s) + c_6\sin(\kappa_o s)$$

Next, we could apply the initial data implicit within the construction of $a(s)$ and $b(s)$ and seek to show $||\alpha(s) - C|| = 1/\kappa_o$. I will stop here as it turns out this proof is needlessly complicated. $\square$

The basic idea of the proof above is often successful. And, sometimes, we have no choice but to work through a problem in differential equations paired with difficult algebra. But, the proof which follows is certainly an easier approach. Following O'neill:

Proof 2: Let α be a unit-speed, non-linear curve with constant curvature $\kappa(s) = \kappa_o \neq 0$ and $\tau(s) = 0$ for all $s \in I$. Consider, at any point along the curve, the normal vector should point towards the center C of the conjectured circle. That is, $C - RN(s) = \alpha(s)$ for $R = 1/\kappa_o$. Define

$$\gamma(s) = \alpha(s) + RN(s)$$

differentiating we obtain:

$$\gamma'(s) = \alpha'(s) + RN'(s) = T(s) - R\kappa_o T(s) = T(s) - T(s) = 0$$

therefore $\gamma'(s) = 0$ for all $s \in I$. Let $s_o \in I$ and observe $\gamma(s_o) = \alpha(s_o) + RN(s_o) = \gamma(s)$ for all $s \in I$. Thus identify $C = \alpha(s_o) + RN(s_o)$ is the fixed center of the circle. Hence $\alpha(s) - C = RN(s)$ for all $s \in I$ and so $||\alpha(s) - C|| = R$ for all $s \in I$. Therefore, α is on a circle of radius $R = 1/\kappa_o$. $\square$

Example 2.4.7. *If a curve is on a sphere then it is at least as curved as a great circle on the sphere. To see this, consider $\alpha : I \to \mathbb{R}^3$ a unit-speed regular curve on the sphere with center C and radius R. We are given $||\alpha(s) - C|| = R$ for all $s \in I$. Of course,*

$$(\alpha - C) \bullet (\alpha - C) = R^2$$

hence, as $\alpha' = T$, the product rule and commutativity of the dot-product yield

$$2T \bullet (\alpha - C) = 0.$$

Divide by two and differentiating once more,

$$T' \bullet (\alpha - C) + T \bullet T = 0.$$

But, $T' = \kappa N$ and $T \bullet T = 1$ thus

$$\kappa N \bullet (\alpha - C) + 1 = 0 \quad \Rightarrow \quad -1/\kappa = N \bullet (\alpha - C)$$

thus, by Cauchy-Schwarz inequality and facts $||N|| = 1$ and $||\alpha - C|| = R$,

$$\frac{1}{\kappa} \leq R$$

Therefore, $\kappa \geq 1/R$. The smallest curvature possible for a curve on a sphere is the case the curve is a great circle. A great circle on a sphere of radius R is naturally a circle of radius R and given our previous work on the helix example we know $\kappa = 1/R$ for the circle.

2.4.1 the non unit-speed case

Up to this point, we have studied curves with unit-speed. In other words, we have thus far studied the Frenet Serret theory for arclength parametrized curves. There are curves for which no such parameterization can be reasonable expressed, yet, relatively simple formulas are known for variable speed parameterizations.

Consider $\alpha : I \to \mathbb{R}^3$ a regular smooth parameterized curve. Also, let $\bar{\alpha} : J \to \mathbb{R}^3$ be the reparametrization of α by the arclength function $s : I \to J$. In particular, $\alpha(t) = \bar{\alpha}(s(t))$ for each $t \in I$. If we define $\bar{T}, \bar{N}, \bar{B}, \bar{\kappa}$ and $\bar{\tau}$ as discussed earlier in this section then T, N, B, κ and τ in the t-domain are defined by reparametrization:

$$T(t) = \bar{T}(s(t)), \quad N(t) = \bar{N}(s(t)), \quad B(t) = \bar{B}(s(t)), \quad \kappa(t) = \bar{\kappa}(s(t)), \quad \tau(t) = \bar{\tau}(s(t)).$$

We denote the speed by v where $v = \frac{ds}{dt} = ||\alpha'(t)||$. The chain-rule yields:

$$\alpha'(t) = \frac{d}{dt}\left[\bar{\alpha}(s(t))\right] = \bar{\alpha}'(s(t))\frac{ds}{dt} = v\bar{T}(s(t)) \qquad \star$$

likewise,

$$T'(t) = \frac{d}{dt}\left[\bar{T}(s(t))\right] = \bar{T}'(s(t))\frac{ds}{dt} = v(t)\bar{\kappa}(s(t))\bar{N}(s(t)) = v(t)\kappa(t)N(t).$$

Thus, the Frenet Serret equation for a non-unit-speed curve is modified to $\dfrac{dT}{dt} = v\kappa N$. The same argument applies to the derivative of N and B hence:

Theorem 2.4.8. *Frenet Serret Equations for non-unit speed curves.*

Let α be a non-linear regular smooth curve with speed $v = ||\alpha'(t)||$ and T, N, B, κ and τ as defined through the unit-speed reparameterization then

$$\frac{dT}{dt} = v\kappa\, N$$
$$\frac{dN}{dt} = -v\kappa\, T + v\tau\, B$$
$$\frac{dB}{dt} = -v\tau N$$

Moreover, $\kappa = \frac{1}{v}||T'||$ and $\tau = -\frac{1}{v}B' \bullet N$.

Proof: we discussed how the chain-rule yields the modified Frenet Serret equations. It remains to prove the formulas for curvature and torsion in the t-domain: note,

$$||T'|| = ||v\kappa N|| = v\kappa \quad \Rightarrow \quad \kappa = \frac{1}{v}||T'||.$$

Also, the dot-product of $B' = -v\tau N$ by N to obtain $B' \bullet N = -v\tau N \bullet N$ thus $\tau = -\frac{1}{v}B' \bullet N$. $\square$

I should emphasize, if you wish to calculate curvature, torsion and the Frenet Serret frame for a non-unit speed curve then you must take care to add the appropriate speed factors. Let us return to $\star$ to examine how the speed factor plays into the acceleration:

$$\alpha'(t) = vT \quad \Rightarrow \quad \alpha''(t) = \frac{dv}{dt}T + vT' = \frac{dv}{dt}T + v^2\kappa N$$

Therefore, we find the acceleration vector is orthogonal to the binormal vector. The tangential component of α'' is $\alpha'' \bullet T = \frac{dv}{dt}$ which describes how the curve is speeding up at the given time. On the other hand, the normal component of α is $\alpha'' \bullet N = \kappa v^2$. Notice, the radius of curvature R is related to the curvature by $\kappa = 1/R$ so we can write the normal component of the acceleration instead as v^2/R. This normal component is directed towards the center of the osculating circle. If you had university physics then you should recognize the results as the tangential and centripetal acceleration formulas. Perhaps you recall the use of $a = \sqrt{a_T^2 + a_N^2} = \sqrt{(dv/dt)^2 + v^4/R^2}$ to solve problems such as a car accelerating around a turn while changing its speed.

Proposition 2.4.9. *acceleration in terms of curvature and speed*

Let α be a non-linear regular smooth curve with speed $v = ||\alpha'(t)||$ then $\alpha'' = \frac{dv}{dt}T + \kappa v^2 N$

Example 2.4.10. *Suppose $\alpha(t) = (t, t^2, t^2)$. Calculate the Frenet apparatus or at least try.*

$$\alpha'(t) = U_1 + 2tU_2 + 2tU_3 \quad \Rightarrow \quad v = \sqrt{1 + 8t^2} \quad \Rightarrow \quad T(t) = \frac{U_1 + 2tU_2 + 2tU_3}{\sqrt{1 + 8t^2}}$$

Thus,

$$\begin{aligned} T'(t) &= \frac{2U_2 + 2U_3}{\sqrt{1+8t^2}} - \frac{1}{2}\frac{U_1 + 2tU_2 + 2tU_3}{(\sqrt{1+8t^2})^3}(16t) \\ &= \frac{2(1+8t^2)(U_2+U_3) - 8t(U_1 + 2tU_2 + 2tU_3)}{(\sqrt{1+8t^2})^3} \\ &= \frac{-8tU_1 + 2U_2 + 2U_3}{(\sqrt{1+8t^2})^3} \\ &= \frac{2}{(\sqrt{1+8t^2})^3}\big(-4tU_1 + U_2 + U_3\big) \end{aligned}$$

Thus,

$$\kappa(t) = \frac{1}{v}||T'(t)|| = \frac{1}{\sqrt{1+8t^2}}\frac{2\sqrt{16t^2+2}}{(\sqrt{1+8t^2})^3} \quad \Rightarrow \quad \kappa(t) = \frac{2\sqrt{16t^2+2}}{(1+8t^2)^2}.$$

I leave the calculation of $B(t)$ and the torsion to the reader. Also, I invite the reader to verify that application of Theorem 2.4.9 to the acceleration of the given curve:

$$\alpha'' = 2U_2 + 2U_3 = \frac{-8t}{(\sqrt{1+8t^2})^3}\,T(t) + \frac{2\sqrt{16t^2+2}}{1+8t^2}\,N(t)$$

You can see problems such as the last example are good candidates for CAS assistance. Also, the theorem below may be helpful. You may have noticed I already assumed we could calculate $T(t)$ as given below:

Theorem 2.4.11. *slick formulas for the Frenet apparatus*

Let α be a non-linear regular smooth curve with speed $v = ||\alpha'(t)||$ and T, N, B, κ and τ as defined through the unit-speed reparameterization then

$$T = \frac{\alpha'}{||\alpha'||}, \quad B = \frac{\alpha' \times \alpha''}{||\alpha' \times \alpha''||}, \quad N = B \times T,$$

$$\kappa = \frac{||\alpha' \times \alpha''||}{||\alpha'||^3}, \quad \tau = \frac{(\alpha' \times \alpha'') \bullet \alpha'''}{||\alpha' \times \alpha''||^2}$$

Proof: see pages 72-73 of O'neill. $\square$

O'neill also give a nice example which showcases how these formulas work on page 73. In addition, O'neill proves part of the following assertions. I leave these to the reader, but, I thought including them would be wise. It is likely I prove part of this in lecture.

Theorem 2.4.12. *curves characterized by curvature and torsion*

Let α be a non-linear regular smooth curve then

(i.) $\kappa = 0$ iff the curve is part of a straight line

(ii.) $\tau = 0$ iff the curve is a planar curve

(iii.) $\kappa > 0$ constant and $\tau = 0$ iff the curve is part of a circle

(iv.) $\kappa > 0$ constant and $\tau > 0$ constant iff the curve is part of a circular helix

(v.) τ/κ nonzero and constant iff the curve is part of a cylindrical helx

Proof: (i.) follows from part (ii.) of Theorem 2.3.8, we also proved (ii.) in Theorem 2.4.5, and we proved (iii.) in Theorem 2.4.6. Parts of the proofs of (iv.) and (v.) can be found in O'neill. $\square$

2.5 covariant derivatives

The covariant derivative of a vector field with respect to another vector field yields a new vector field which describes the change in a given vector field along the direction of the second field. We use $\mathfrak{X}(\mathbb{R}^3)$ to denote smooth vector fields on $\mathbb{R}^3$.

Definition 2.5.1. *covariant derivative*

Suppose $W \in \mathfrak{X}(\mathbb{R}^3)$ and $v \in T_p\mathbb{R}^3$. The **covariant derivative** of W with respect to v at p is the tangent vector:

$$(\nabla_v W)(p) = W(p + tv)'(0) \in T_p\mathbb{R}^3.$$

If $V \in \mathfrak{X}(\mathbb{R}^3)$ and $V(p) = v_p$ then the assignment $p \to (\nabla_{v_p} W)(p)$ defines $\nabla_V W \in \mathfrak{X}(\mathbb{R}^3)$ and we say $\nabla_V W$ is the **covariant derivative** of W with respect to V.

The covariant derivative above is essentially just a directional derivative, but, the wrinkle is that W is not just a real-valued function. Since W is a vector field there are 3 components which can change and the change of W in the v-direction is described by the change of all three components of W in tandem.

Example 2.5.2. *If $W(p) = aU_1 + bU_2 + cU_3$ for constants $a, b, c \in \mathbb{R}$ for all $p \in \mathbb{R}^3$ then $W(p+tv) = aU_1 + bU_2 + cU_3$ hence $W'(p + tv) = 0$ thus $(\nabla_v W)(p) = 0$ for all $p \in \mathbb{R}^3$ hence $\nabla_V W = 0$ for any choice of $V \in \mathfrak{X}(\mathbb{R}^3)$ as the calculation held for arbitrary v at each p.*

Example 2.5.3. *What about the change of $W = x^2U_1 + yU_3$ along $v = 2U_2 + U_3$ at $p = (1, 2, 3)$? Calculate,*

$$W(p + tv) = W(1 + 2t, 2, 3 + t) = (1 + 2t)^2U_1 + (3 + t)U_3$$

thus,

$$W'(p + tv) = 4(1 + 2t)U_1 + U_3 \quad \Rightarrow \quad W'(p + tv)(0) = 4U_1 + U_3.$$

Therefore, $(\nabla_v W)(1, 2, 3) = 4U_1 + U_3$.

The definition of the directional derivative in terms of a parametrized line soon gave way to a more efficient formula in terms of the gradient, the same happens here:

Proposition 2.5.4. *coordinate derivative formula for covariant derivative*

Let $V, W \in \mathfrak{X}(\mathbb{R}^3)$ then $\nabla_V W = \sum_{j=1}^{3} V[W^j]U_j$.

Proof: let $V(p) = v$ and recall Equation 1.1 to go from line two to line 3,

$$\begin{aligned} W'(p+tv)(0) &= \sum_{j=1}^{3} \frac{dW^j(p+tv)}{dt}(0)\, U_j \\ &= \sum_{j=1}^{3} (DW^j)(v)(p) U_j \\ &= \sum_{j=1}^{3} v[W^j]\, U_j \end{aligned}$$

Therefore, as this holds at each $p \in \mathbb{R}^3$, we conclude $\nabla_V W = \sum_{j=1}^{3} V[W^j]U_j$. $\square$

To see why I claim the result above has something to do with the gradient we can expand the formula further into components of $V = \sum_i V^i U_i$ where $U_i = \partial_i$ in case you forgot. Observe:

$$\nabla_V W = \sum_{j=1}^{3} V[W^j]U_j = \sum_{j=1}^{3} \sum_{i=1}^{3} V^i \partial_i[W^j]U_j = \sum_{j=1}^{3} \left(V \bullet \nabla W^j\right)\, U_j \tag{2.2}$$

where $\nabla W^j = (\partial_1 W^j)U_1 + (\partial_2 W^j)U_2 + (\partial_3 W^j)U_3$. Equation 2.2 implies the following properties: the covariant derivative $\nabla_V W$ is additive in both V and W. However, homogeneity of the V-argument allows for us to extract scalar functions whereas homogeneity of W allows only for constants. Of course, there is also a product rule tied to the W entry. Let us be precise:

Proposition 2.5.5. *properties of the covariant derivative on* $\mathbb{R}^3$

Let $U, V, W \in \mathfrak{X}(\mathbb{R}^3)$ then

(i.) $\nabla_{U+V} W = \nabla_U W + \nabla_V W$

(ii.) $\nabla_V (U + W) = \nabla_V U + \nabla_V W$

(iii.) $\nabla_{fV} W = f \nabla_V W$ for all smooth $f : \mathbb{R}^3 \to \mathbb{R}$

(iv.) $\nabla_V (cW) = c \nabla_V W$ for all $c \in \mathbb{R}$

(v.) $\nabla_V (fW) = V[f]\, W + f \nabla_V W$ for all smooth $f : \mathbb{R}^3 \to \mathbb{R}$

(vi.) $U[V \bullet W] = \nabla_U V \bullet W + V \bullet \nabla_U W$

Proof: for (i.) and (iii.) consider $U, V, W \in \mathfrak{X}(\mathbb{R}^3)$ and $f : \mathbb{R}^3 \to \mathbb{R}$ smooth. Use Equation 2.2

$$\begin{aligned}
\nabla_{U+fV} W &= \sum_{j=1}^{3} \big((U + fV) \bullet \nabla W^j\big)\, U_j \\
&= \sum_{j=1}^{3} \big(U \bullet \nabla W^j + fV \bullet \nabla W^j\big)\, U_j \\
&= \sum_{j=1}^{3} \big(U \bullet \nabla W^j\big)\, U_j + f \sum_{j=1}^{3} \big(V \bullet \nabla W^j\big)\, U_j \\
&= \nabla_U W + f \nabla_V W.
\end{aligned}$$

The proof of (ii.) and (iv.) and (v.) is very similar. Consider, U, V, W and f as in the proposition and use Proposition 2.5.4 subject the observation $(U + fW)^j = U^j + fW^j$ by defn. of vector add.,

$$\begin{aligned}
\nabla_V (U + fW) &= \sum_{j=1}^{3} V[U^j + fW^j]\, U_j \\
&= \sum_{j=1}^{3} \big(V[U_j] + V[fW^j]\big)\, U_j \\
&= \sum_{j=1}^{3} \big(V[U_j] + V[f]W^j + fV[W^j]\big)\, U_j \\
&= \sum_{j=1}^{3} V[U_j]\, U_j + V[f] \sum_{j=1}^{3} W^j\, U_j + f \sum_{j=1}^{3} V[W^j]\, U_j \\
&= \nabla_V U + V[f]\, W + f \nabla_V W.
\end{aligned}$$

I made use of Proposition 1.2.7 in the calculation above. Essentially, that proposition follows almost directly from the corresponding properties for the coordinate partial derivatives.

Finally, consider the following proof of property (vi.). Let $U, V, W \in \mathfrak{X}(\mathbb{R}^3)$. Proposition from Proposition 2.5.4 gives

$$\nabla_V W = \sum_{j=1}^{3} V[W^j] U_j \quad \Rightarrow \quad (\nabla_V W)^i = V[W^i]$$

We calculate by Proposition 1.2.7 once more,

$$\begin{aligned}
U[V \bullet W] &= U\left[\sum_{i=1}^{3} V^i W^i\right] \\
&= \sum_{i=1}^{3} \big(U[V^i]\, W^i + V^i\, U[W^i]\big) \\
&= \sum_{i=1}^{3} \big((\nabla_U V)^i\, W^i + V^i\, (\nabla_U W)^i\big) \\
&= (\nabla_U V) \bullet W + V \bullet (\nabla_U W). \qquad \square
\end{aligned}$$

Of course, all the calculations above are to be done at a point p, but, as the identity holds generally we obtain statements about functions and vector fields. I decided the proofs without explicit points are easier to follow. Perhaps the same is true for examples: incidentally, it is useful to write the formula for the covariant derivative in explicit detail for the sake of the example which follows:

$$\nabla_V W = V[W^1]U_1 + V[W^2]U_2 + V[W^3]U_3.$$

Example 2.5.6. *Let $V = xU_1 + y^2U_2 + z^3U_3$ and $W = yzU_1 + xyU_3$. Recall our notation U_1, U_2, U_3 masks the fact that these are derivations; $U_1 = \partial_x$, $U_2 = \partial_y$ and $U_3 = \partial_z$ thus:*

$$V[yz] = x\partial_x[yz] + y^2\partial_y[yz] + z^3\partial_z[yz] = y^2z + z^3y.$$
$$V[xy] = x\partial_x[xy] + y^2\partial_y[xy] + z^3\partial_z[xy] = xy + y^2x.$$

Therefore, as $W^1 = yz$ and $W^3 = xy$ we find

$$\nabla_V W = V[yz]U_1 + V[xy]U_3 = (y^2z + z^3y)U_1 + (xy + y^2x)U_3.$$

Example 2.5.7. *Calculate $\nabla_V V$. To calculate $\nabla_V V$ it may be instructive to use property (vi.),*

$$V[V \bullet V] = (\nabla_V V) \bullet V + V \bullet (\nabla_V V).$$

Thus, as $V[f^2] = 2fV[f]$ for smooth f we have:

$$(\nabla_V V) \bullet V = \frac{1}{2}V[||V||^2] = ||V||\, V[||V||]$$

Divide by $||V||$ to see the component of $\nabla_V V$ in the V-direction is simply $V[||V||]$. As a particular application of this calculation, notice if the vector field is of constant length then $V[||V||] = 0$ which means that V does not change in the $V(p)$ direction at p.

Finally, a word of caution, the idea of covariant differentiation shown in this section is just the first chapter in a larger story. More generally, the covariant derivative is tied to something called a *connection on a fiber bundle*. Roughly, this extends the idea of distant parallelism to general spaces. I suppose I should also mention, in General Relativity there is a covariant derivative with respect to the metric connection which is used define *geodesics*. General relativity claims particles following geodesics in the spacetime manifold explains gravity in nature. In any event, for now we study the covariant derivative in $\mathbb{R}^3$. We later study a different covariant derivative associated with the calculus of a surface. Next, we continue to study the covariant derivative, but, with other possible non-Cartesian frames in $\mathbb{R}^3$.

2.6 frames and connection forms

The covariant derivative of a vector field in Cartesian coordinate simply involves partial derivatives of coordinates. In this section we study how to calculate $\nabla_V W$ with respect to a non-constant frame for $\mathbb{R}^3$. In addition to the derivative terms we saw in the last section there are new terms corresponding to nontrivial *connection coefficients*. The connection coefficients of a given frame then allow us to somewhat indirectly study the geometry of objects to which the frame is naturally fit. For example, the curvature and torsion played the role of connection coefficients for the curve to which we fit the T, N, B frame. However, that is not quite the right picture as we intend to fit frames to surfaces. We will explain in this section that the connection coefficients are naturally identified with a matrix of one-forms. Ultimately, the exterior calculus of these matrix-valued one-forms

If $\{E_1, E_2, E_3\}$ is a frame for $\mathbb{R}^3$ and $V \in \mathfrak{X}(\mathbb{R}^3)$ then there exist functions f^1, f^2, f^3 called the **components of** V with respect to the frame $\{E_1, E_2, E_3\}$

$$V = f^1E_1 + f^2E_2 + f^3E_3$$

Orthonormality of the frame shows the component functions are calculated from:

$$f^j = V \bullet E_j$$

Suppose $V, W \in \mathfrak{X}(\mathbb{R}^3)$ where $V = \sum f^iE_i$ and $W = \sum_j g^jE_j$. Calculate, by Proposition 2.5.5,

$$\begin{aligned}
\nabla_V W &= \nabla_{\sum f^iE_i}\Big(\sum_j g^jE_j\Big) \\
&= \sum_{i,j} f^i\nabla_{E_i}\big(g^jE_j\big) \\
&= \sum_{i,j} f^i\,\big[\,E_i[g^j]E_j + g^j\nabla_{E_i}(E_j)\,\big] \\
&= \underbrace{\sum_j V[g^j]E_j}_{\text{I.}} + \underbrace{\sum_{i,j} f^ig^j\nabla_{E_i}(E_j)}_{\text{II.}}
\end{aligned}$$

Recall, in terms of the Cartesian frame U_1, U_2, U_3 if $V = \sum_i V^iU_i$ and $W = \sum_j W^jU_j$ then $\nabla_V W = \sum_j V[W^j]U_j$ which resembles term (I.). In comparison, there is no type-II. term as should be expected since U_i is contant hence $\nabla_{U_i}(U_j) = 0$.

Definition 2.6.1. *connection forms*[6]

If E_1, E_2, E_3 is a frame for $\mathbb{R}^3$ then define $\omega_{ij}(p) \in (T_p\mathbb{R}^3)^*$ by

$$\omega_{ij}(v) = (\,\nabla_v E_i\,) \bullet E_j(p)$$

for each $v \in T_p\mathbb{R}^3$. That is, ω_{ij} is a differential one-form on $\mathbb{R}^3$ defined by the assignment $p \mapsto \omega_{ij}(p)$ for each $p \in \mathbb{R}^3$.

Half of the proposition below is a simple consquence of Equation 2.1.

[6]I was tempted to introduce **structure functions** $C^k_{ij} = E_k \bullet \nabla_{E_i}(E_j)$, but, I behave. See page 202-203 of *Manifolds, Tensors and Forms: An Introduction for Mathematicians and Physicsts* by Renteln if you wish to read more in that direction. In particular, he relates the structure functions to the Christoffel symbols. This approach

Proposition 2.6.2. *properties of the covariant derivative on* $\mathbb{R}^3$

Let $\{E_1, E_2, E_3\}$ be a frame on $\mathbb{R}^3$ then $\omega_{ij} = -\omega_{ji}$ and $\nabla_V E_i = \displaystyle\sum_{j=1}^{3} \omega_{ij}(V)\, E_j$.

Proof: By a slight modification of Example 2.5.7, consider:

$$V[E_i \bullet E_j] = (\nabla_V E_i) \bullet E_j + E_i \bullet (\nabla_V E_j) \quad \Rightarrow \quad 0 = \omega_{ij}(V) + \omega_{ji}(V)$$

for all V hence $\omega_{ij} = -\omega_{ji}$. Next, note $\omega_{ij}(V) = (\nabla_V E_i) \bullet E_j$ is the j-th component of $\nabla_V E_i$ in the $\{E_1, E_2, E_3\}$ frame. If $W = \nabla_V E_i$ then $W = \sum_j (W \bullet E_j) E_j$. Therefore, $\nabla_V E_i = \displaystyle\sum_{j=1}^{3} \omega_{ij}(V)\, E_j$. $\square$

It may be useful to express these explicitly:

$$\begin{aligned}
\nabla_V E_1 &= \omega_{12}(V) E_1 + \omega_{13}(V) E_3 \\
\nabla_V E_2 &= -\omega_{12}(V) E_1 + \omega_{23}(V) E_3 \\
\nabla_V E_3 &= -\omega_{13}(V) E_1 - \omega_{23}(V) E_2
\end{aligned}$$

We can arrange the differential forms ω_{ij} into a matrix ω

$$\omega = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix}$$

This is a *matrix* of one-forms. Let us pause to develop the theory of matrices of forms over $\mathbb{R}^n$

2.6.1 on matrices of differential forms

O'neill does not write $\wedge$ for $A \wedge B$ I write below. But, I'd rather write a few wedges to emphasize the nature of the calculation. Essentially, the definition below just replaces the ordinary product of real numbers in the usual matrix algebra with a wedge product of forms.

Definition 2.6.3. *exterior algebra and calculus of matrices of forms:*

Let A_{ik} be a p-form and B_{kj} be a q-form for $1 \leq i \leq m$, $1 \leq k \leq r$ and $1 \leq j \leq n$ then we say A is an $m \times r$ matrix of p-forms and B is a $r \times n$ matrix of q-forms. Then we say A and B are **multipliable** and define the $m \times n$-matrix of $(p+q)$-forms $A \wedge B$ by:

$$(A \wedge B)_{ij} = \sum_{k=1}^{r} A_{ik} \wedge B_{kj}.$$

Likewise, we denote the **exterior derivative of** A by dA which is defined to be the $m \times r$-matrix of $(p+1)$-forms given by $(dA)_{ik} = dA_{ik}$.

We also denote, $A = \sum_{i,k} A_{ik} E_{ik}$ and $B = \sum_{k,j} B_{kj} E_{kj}$ where $(E_{ij})_{lm} = \delta_{il}\delta_{jm}$. The **standard-matrix-basis** E_{ij} is at times very useful for proofs and general questions. Furthermore, we define A^T in the usual manner; $(A^T)_{ij} = A_{ji}$. Likewise, addition and substraction of forms as well as multiplication by smooth functions are naturally defined.

Example 2.6.4. *Let* $A = \begin{bmatrix} dx & dy \\ dz & z^2dy + y^2dz \end{bmatrix}$ *and* $B = \begin{bmatrix} dx+dy & 0 \\ z^2dy & dx+dz \end{bmatrix}$ *then*

$$\begin{aligned} A \wedge B &= \begin{bmatrix} dx & dy \\ dz & z^2dy + y^2dz \end{bmatrix} \wedge \begin{bmatrix} dx+dy & 0 \\ z^2dy & dx+dz \end{bmatrix} \\ &= \left[\begin{array}{c|c} dx \wedge (dx+dy) + dy \wedge z^2dy & dx \wedge 0 + dy \wedge (dx+dz) \\ \hline dz \wedge (dx+dy) + (z^2dy+y^2dz) \wedge z^2dy & dz \wedge 0 + (z^2dy+y^2dz) \wedge (dx+dz) \end{array}\right] \\ &= \left[\begin{array}{c|c} dx \wedge dy & -dx \wedge dy + dy \wedge dz \\ \hline dz \wedge dx - dy \wedge dz + y^2z^2dz \wedge dy & z^2(dy \wedge dz - dx \wedge dy) + y^2dz \wedge dx \end{array}\right] \end{aligned}$$

The last step I included just to illustrate one way to simplify the answer. Also, calculate:

$$dA = \begin{bmatrix} 0 & 0 \\ 0 & 2(y-z)dy \wedge dz \end{bmatrix} \qquad \& \qquad dB = \begin{bmatrix} 0 & 0 \\ 2zdz \wedge dy & 0 \end{bmatrix}.$$

Observe, we can reasonably express these as matrix-valued two-forms; $dA = (2(y-z)dy \wedge dz)\, E_{22}$ *and* $dB = (2zdz \wedge dy)\, E_{21}$*.*

Proposition 2.6.5. *product rule for matrices of forms*

Let A be a matrix of p-forms and$B$ be a matrix of q-forms and suppose A, B are multipliable then

$$d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB.$$

Proof: consider,

$$\begin{aligned} d(A \wedge B)_{ij} &= d\left(\sum_k A_{ik} \wedge B_{kj}\right) && : \text{defn. of } A \wedge B \\ &= \sum_k d\left(A_{ik} \wedge B_{kj}\right) && : \text{additivity of } d \\ &= \sum_k \left(dA_{ik} \wedge B_{kj} + (-1)^p A_{ik} \wedge dB_{kj}\right) && : \text{by Prop. 1.3.3} \\ &= \left(dA \wedge B + (-1)^p A \wedge dB\right)_{ij} \end{aligned}$$

Therefore, as the above holds for all i, j, we conclude $d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB$ $\square$

Of course, we probably could spend much more time and effort working out general results about the exterior calculus of forms of matrices. For example, I invite the reader to verify associativity $(A \wedge B) \wedge C = A \wedge (B \wedge C)$. I wil behave and get back on task now. We are mainly interested in the **attitude matrix** of Definition 2.2.11. In Theorem 2.2.12 we learned that the attitude matrix is point-wise orthogonal; $A^TA = I$. A matrix of functions is a matrix of zero-forms and it is included in the proposition above. Moreover, in this special case we can reasonably omit or include the wedge. In particular, perhaps it is helpful to write $A^T \wedge A = I$. From this we obtain the following:

Proposition 2.6.6. *attitude matrix*

Let A be the attitude matrix of a given frame then

$$dA^T \wedge A = -A^T \wedge dA \qquad \& \qquad dA \wedge A^T = -A \wedge dA^T.$$

Moreover, $dA = -A \wedge dA^T \wedge A$.

Proof: if A is an attitude matrix then $A^TA = I$ and $AA^T = I$. But, A and A^T are multipliable matrices of zero-forms hence the product rule for matrices of forms yield

$$dA^T \wedge A + A^T \wedge dA = dI \qquad \& \qquad dA \wedge A^T + A \wedge dA^T = dI.$$

Note I is a constant matrix hence $dI = 0$ and we obtain:

$$dA^T \wedge A = -A^T \wedge dA \qquad \& \qquad dA \wedge A^T = -A \wedge dA^T.$$

Finally, multiply $dA \wedge A^T = -A \wedge dA^T$ by A on the right, I invite the reader to verify $dA \wedge I = dA$ and hence $dA = -A \wedge dA^T \wedge A$. $\square$

We could reasonably write $dA = -A(dA^T)A$ and $(dA^T)A = -A^T(dA)$ as A and A^T are just ordinary matrices of functions. I suppose all of this is a bit abstract for the first course, but, the result we next examine redeems the merit of our discussion. We find that the connection form can be calculated by mere matrix multiplication and exterior differentiation of the attitude matrix!

Theorem 2.6.7. *attitude and the connection form*

Let A be the attitude matrix of a given frame then $\omega = dA \wedge A^T$.

Proof: let E_1, E_2, E_3 be a frame and A_{ij} the components of the attitude matrix:

$$E_i = A_{i1}U_1 + A_{i2}U_2 + A_{i3}U_3 = \sum_{k=1}^{3} A_{ik}U_k$$

we defined $\omega_{ij}(V) = (\nabla_V E_i) \bullet E_j$ for all $V \in \mathfrak{X}(\mathbb{R}^3)$. Thus,

$$\begin{aligned}
\omega_{ij}(V) &= \nabla_V \left(\sum_{k=1}^{3} A_{ik}U_k \right) \bullet \ E_j \\
&= \sum_{k=1}^{3} \nabla_V \left(A_{ik}U_k\right) \bullet \ E_j \\
&= \sum_{k=1}^{3} \left(V[A_{ik}]\, U_k + A_{ik}\nabla_V U_k\right) \bullet \ E_j \\
&= \sum_{k=1}^{3} V[A_{ik}]\, U_k \bullet \left(\sum_{l=1}^{3} A_{jl}U_l \right) \\
&= \sum_{k,l=1}^{3} V[A_{ik}]A_{jl}\delta_{kl} \\
&= \sum_{k=1}^{3} V[A_{ik}]A_{jk} \\
&= (V[A]A^T)_{ij}
\end{aligned}$$

To be clear, $V[A]$ is a matrix of functions and $(V[A])_{ij} = V[A_{ij}] = (dA_{ij})(V) = (dA(V))_{ij}$ or,

$$V[A] = \begin{bmatrix} V[A_{11}] & V[A_{12}] & V[A_{13}] \\ V[A_{21}] & V[A_{22}] & V[A_{23}] \\ V[A_{31}] & V[A_{32}] & V[A_{33}] \end{bmatrix} = \begin{bmatrix} dA_{11}(V) & dA_{12}(V) & dA_{13}(V) \\ dA_{21}(V) & dA_{22}(V) & dA_{23}(V) \\ dA_{31}(V) & dA_{32}(V) & dA_{33}(V) \end{bmatrix} = dA(V)$$

In view of the notation above, we conclude $\omega_{ij} = (dA \wedge A^T)_{ij}$ for all i, j hence $\omega = dA \wedge A^T$. $\square$

I suppose, I should admit, we are defining the evaluation of a matrix of forms on a vector field in the natural manner; $(B(V))_{ij} = B_{ij}(V)$. That is, we evaluate component-wise. It happens that matrix multiplication by a 0-form matrix can be done before or after the evaluation by V hence there is no ambiguity in writing $dA(V)A^T$ or $(dA \wedge A^T)(V)$.

Example 2.6.8. *Following Examples 2.2.9 and 2.2.15, the cylindrical coordinate frame has attitude matrix*

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad dA = \begin{bmatrix} -\sin\theta d\theta & \cos\theta d\theta & 0 \\ -\cos\theta d\theta & -\sin\theta d\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore,

$$\omega = dA \wedge A^T = \begin{bmatrix} -\sin\theta d\theta & \cos\theta d\theta & 0 \\ -\cos\theta d\theta & -\sin\theta d\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & d\theta & 0 \\ -d\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 2.6.9. *Following Examples 2.2.10 and 2.2.16, the spherical coordinate frame has attitude matrix*

$$A = \begin{bmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \quad \& \quad A^T = \begin{bmatrix} \cos\theta\sin\phi & \cos\theta\cos\phi & -\sin\theta \\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix}$$

Thus,

$$dA = \begin{bmatrix} -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \\ -\sin\theta\cos\phi & \cos\theta\cos\phi & 0 \\ -\cos\theta & -\sin\theta & 0 \end{bmatrix} d\theta + \begin{bmatrix} \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ -\cos\theta\sin\phi & -\sin\theta\sin\phi & -\cos\phi \\ 0 & 0 & 0 \end{bmatrix} d\phi$$

Calculate (the product of matrices associated with $d\theta$ in $dA \wedge A^T$):

$$\begin{bmatrix} -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \\ -\sin\theta\cos\phi & \cos\theta\cos\phi & 0 \\ -\cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \cos\theta\sin\phi & \cos\theta\cos\phi & -\sin\theta \\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \sin\phi \\ 0 & 0 & \cos\phi \\ -\sin\phi & -\cos\phi & 0 \end{bmatrix}$$

and (the product of matrices associated with $d\phi$ in $dA \wedge A^T$):

$$\begin{bmatrix} \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ -\cos\theta\sin\phi & -\sin\theta\sin\phi & -\cos\phi \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta\sin\phi & \cos\theta\cos\phi & -\sin\theta \\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore,

$$\omega = dA \wedge A^T = \begin{bmatrix} 0 & d\phi & \sin\phi\, d\theta \\ -d\phi & 0 & \cos\phi\, d\theta \\ -\sin\phi\, d\theta & -\cos\phi\, d\theta & 0 \end{bmatrix}.$$

As much fun as this is, we can do better. See the next section. (see Example 2.7.7 for another angle on how to do this calculation).

2.7 coframes and the Structure Equations of Cartan

As we compare a real vector space V and its dual V^* it is often interesting to pair a basis $\beta = \{v_1, \dots, v_n\}$ of V with a *dual-basis* $\{v^1, \dots, v^n\}$ for V^*. In particular, we required $v^i(v_j) = \delta_{ij}$ for all i, j. We now introduce the same construction for frames on $\mathbb{R}^3$, but, as this is done for arbitary $p \in \mathbb{R}^3$ we have for a given frame of vector fields a coframe of differential one forms.

Definition 2.7.1. *coframe on $\mathbb{R}^3$:*

Suppose $\{E_1, E_2, E_3\}$ is a frame on $\mathbb{R}^3$ then we say a set of differential one-forms $\theta^1, \theta^2, \theta^3$ on $\mathbb{R}^3$ is a **coframe** if $\theta^i(E_j) = \delta_{ij}$ for all i, j.

Left to my own devices, I'd probably use $\theta^i = E^i$, but, I'll try to stick close to O'neill's notation[7]. To say θ^j is a differential one-form implies linearity $\theta^j(cV + W) = c\theta^j(V) + \theta^j(W)$ thus, for $V = \sum_i f^i E_i$ we have:

$$\theta^j(V) = \theta^j(\sum_i f^i E_i) = \sum_i f^i \theta^j(E_i) = f^j = E_j \bullet V. \tag{2.3}$$

Example 2.7.2. *The frame $U_1 = \partial_x, U_2 = \partial_y, U_3 = \partial_z$ has coframe $\theta^1 = dx, \theta^2 = dy, \theta^3 = dz$ as*

$$dx^i(\partial_j) = \partial_j x^i = \delta_{ij}.$$

Notice $dx^j(V) = dx^j(V^1U_1 + V^2U_2 + V^3U_3) = V^j$. Thus, $dx^j(V) = V \bullet U_j$.

Proposition 2.7.3. *components with respect to frame and coframe*

If E_1, E_2, E_3 is a frame with coframe $\theta^1, \theta^2, \theta^3$ if $Y \in \mathfrak{X}(\mathbb{R}^3)$ and $\alpha \in \Lambda^1(\mathbb{R}^3)$ then

$$Y = \sum_{j=1}^{3} \theta^j(Y)E_j \qquad \& \qquad \alpha = \sum_{j=1}^{3} \alpha(E_j)\theta^j.$$

Proof: since $E_j(p)$ and $\theta^j(p)$ form bases at each $p \in \mathbb{R}^3$ it follows there must exist functions a^j and b_j for $j = 1, 2, 3$ such that

$$Y = \sum_{j=1}^{3} a^j E_j \qquad \& \qquad \alpha = \sum_{j=1}^{3} b_j \theta^j$$

then notice, $\theta^i(Y) = \theta^i\left(\sum_{j=1}^{3} a^j E_j\right) = \sum_{j=1}^{3} a^j \theta^j(E_j) = a^i$ and $\alpha(E_i) = \sum_{j=1}^{3} b_j \theta^j(E_i) = b_i$. $\square$

The attitude matrix A was defined to be the coefficients A_{ij} for which $E_i = \sum_j A_{ij} U_j$. Naturally, $A_{ij} = E_i \bullet U_j$ by othonormality of the standard Cartesian frame. Now, by Equation 2.3 note:

$$A_{ij} = E_i \bullet U_j = dx^j(E_i)$$

[7]not quite the same, I at least insist the index on the coframe be up since it is a dual basis. On that comment, perhaps I should admit, some authors do replace ω_{ij} with ${\omega_i}^j$ for the sake of matching index positions.

Consider, again, by Equation 2.3 for the second equality,

$$\theta^i = \sum_j \theta^i(U_j)dx^j = \sum_j (E_i \bullet U_j)dx^j = \sum_j A_{ij}dx^j.$$

Thus we have shown the attitude matrix also shows how coframes are related. Let us record this result for future reference:

Proposition 2.7.4. *attitude of coframe*

If E_1, E_2, E_3 is a frame with coframe $\theta^1, \theta^2, \theta^3$ and U_1, U_2, U_3 is the Cartesian frame with coframe dx^1, dx^2, dx^3 on $\mathbb{R}^3$ then $E_i = \sum_j A_{ij}U_j \quad \Leftrightarrow \quad \theta^i = \sum_j A_{ij}dx^j$.

It is useful to use matrix notation to think about a column of one-forms, we can restate the proposition above as follows:

$$\theta = \begin{bmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} \qquad \& \qquad d\xi = \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} \quad \Rightarrow \quad \theta = Ad\xi.$$

Just to be explicit,

$$Ad\xi = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} = \begin{bmatrix} A_{11}dx^2 + A_{12}dx^2 + A_{13}dx^3 \\ A_{21}dx^2 + A_{22}dx^2 + A_{23}dx^3 \\ A_{31}dx^2 + A_{32}dx^2 + A_{33}dx^3 \end{bmatrix} = \begin{bmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} = \theta.$$

With the above notation settled, we can easily derive Cartan's Equations of Structure which directly relate the coframe and connection form through exterior calculus and algebra alone:

Theorem 2.7.5. *Cartan's Structure Equations for* $\mathbb{R}^3$

If E_i is a frame with coframe θ^i and ω is the connection form for the given frame then

$$(\mathbf{i}.)\ \ d\theta^i = \sum_j \omega_{ij} \wedge \theta^j \qquad\qquad (\mathbf{ii}.)\ \ d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}.$$

Proof: structure equation (i.) can be stated as $d\theta = \omega \wedge \theta$ in matrix notation. We derived $\omega = dA \wedge A^T$ in Theorem 2.6.7. Also, recall $A \wedge A^T = I$. Thus consider:

$$\theta = Ad\xi \quad \Rightarrow \quad d\theta = dA \wedge d\xi = dA \wedge A^T A d\xi = \omega \wedge \theta.$$

To prove (ii.) begin by noting and structure equation (ii.) is simply $d\omega = \omega \wedge \omega$ in the matrix of forms notation. Propostion 2.6.5 gives us the product rule:

$$d\omega = d(dA \wedge A^T) = d(dA) \wedge A^T - dA \wedge dA^T = -dA \wedge dA^T \quad \star$$

where the negative sign stems from the fact dA is a matrix of one-forms and $d(dA) = 0$ for the reasons we discussed in the previous chapter. Proposition 2.6.6 suggests $dA^T = -A^T(dA)A^T$ hence replace dA^T in $\star$ to obtain:

$$d\omega = -dA \wedge dA^T = dA \wedge A^T(dA)A^T = (dA \wedge A^T) \wedge (dA \wedge A^T) = \omega \wedge \omega$$

where the next to last step is not entirely necessary, you could do well with less $\wedge$'s. $\square$

The structure equations suggest we can calculate the exterior derivatives of the coframe and connection *without* differentiation.

Example 2.7.6. *Following Examples 2.2.9, 2.2.15 and 2.6.8 consider for the cylindrical frame:*

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \& \qquad \omega = \begin{bmatrix} 0 & d\theta & 0 \\ -d\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

I can read from A that the coframe and frame are:

$$\begin{aligned} \theta^1 &= \cos\theta\, dx + \sin\theta\, dy \\ \theta^2 &= -\sin\theta\, dx + \cos\theta\, dy \\ \theta^3 &= dz. \end{aligned} \qquad \& \qquad \begin{aligned} E_1 &= \cos\theta\, U_1 + \sin\theta\, U_2 = \partial_r \\ E_2 &= -\sin\theta\, U_1 + \cos\theta\, U_2 = r\partial_\theta \\ E_3 &= U_3. \end{aligned}$$

Note $x = r\cos\theta$ and $y = r\sin\theta$ imply $dx = \cos\theta\, dr - r\sin\theta d\theta$ and $dy = \sin\theta\, dr + r\cos\theta d\theta$. Thus,

$$\theta^1 = \cos\theta\,[\cos\theta\, dr - r\sin\theta d\theta] + \sin\theta\,[\sin\theta\, dr + r\cos\theta d\theta] = dr.$$

This is good, $\theta^1(E_1) = dr(\partial_r) = 1$. Likewise, we can derive $\theta^2 = rd\theta$ and $\theta^3 = dz$. Therefore,

$$\omega \wedge \theta = \begin{bmatrix} 0 & d\theta & 0 \\ -d\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} dr \\ rd\theta \\ dz \end{bmatrix} = \begin{bmatrix} -rd\theta \wedge d\theta \\ -d\theta \wedge dr \\ 0 \end{bmatrix}$$

Of course $d\theta \wedge d\theta = 0$ hence $(\omega\wedge\theta)_1 = (\omega\wedge\theta)_3 = 0$ and $(\omega\wedge\theta)_2 = dr \wedge d\theta$. Naturally,

$$d\theta^1 = d\theta^3 = 0 \qquad \& \qquad d\theta^2 = d(rd\theta) = dr \wedge d\theta.$$

So, we have confirmed the first structure equation of Cartan.

Another use of the structure equations is to derive the connection form from the coframe in a subtle manner. In the next example I begin with the coframe of the spherical coordinate system following Examples 2.2.10, 2.2.16 and 2.6.9.

Example 2.7.7. *The coframe to the spherical frame introduced in Example 2.2.10 is given by:*

$$\theta^1 = d\rho, \qquad \theta^2 = \rho\, d\phi, \qquad \theta^3 = \rho\sin\phi\, d\theta$$

Take exterior derivatives:

$$\begin{aligned} d\theta^1 &= 0, \\ d\theta^2 &= d\rho \wedge d\phi, \\ d\theta^3 &= \sin\phi\, d\rho \wedge d\theta + \rho\cos\phi\, d\phi \wedge d\theta. \end{aligned}$$

Hence, the first structure equations yield:

$$\begin{aligned} (I.)\quad d\theta^1 &= \omega_{12}\wedge\theta^2 + \omega_{13}\wedge\theta^3 = \omega_{12}\wedge(\rho\, d\phi) + \omega_{13}\wedge(\rho\sin\phi\, d\theta) = 0 \\ (II.)\quad d\theta^2 &= -\omega_{12}\wedge\theta^1 + \omega_{23}\wedge\theta^3 = -\omega_{12}\wedge d\rho + \omega_{23}\wedge(\rho\sin\phi\, d\theta) = d\rho\wedge d\phi, \\ (III.)\quad d\theta^3 &= -\omega_{13}\wedge\theta^1 - \omega_{23}\wedge\theta^2 = -\omega_{13}\wedge d\rho - \omega_{23}\wedge(\rho\, d\phi) = \sin\phi\, d\rho\wedge d\theta + \rho\cos\phi\, d\phi\wedge d\theta. \end{aligned}$$

We can see from these equations that:

$$\omega_{12} = d\phi, \quad \omega_{13} = \sin\phi\, d\theta, \quad \omega_{23} = \cos\phi\, d\theta \quad \Rightarrow \quad \omega = \begin{bmatrix} 0 & d\phi & \sin\phi\, d\theta \\ -d\phi & 0 & \cos\phi\, d\theta \\ -\sin\phi\, d\theta & -\cos\phi\, d\theta & 0 \end{bmatrix}$$

If you compare with Example 2.6.9 then you can see we agree.

Chapter 3

euclidean geometry

The study of euclidean geometry is greatly simplified by the use of analytic geometry. In particular, $\mathbb{R}^3$ provides a *model* in which the axioms of Euclid are easily realized for lines and points. The exchange of axioms for equations and linear algebra is not a fair trade. Certainly the modern student has a great advantage over those ancients who had mere compass and straightedge constructive geometry. Since the time of Descartes we have identified euclidean geometry with the study of points in $\mathbb{R}^2$ or $\mathbb{R}^3$. We continue this tradition of analytic geometry in this chapter.

Given we take $\mathbb{R}^n$ as euclidean space, we begin by defining the distance between points in terms of the norm and dot product in $\mathbb{R}^n$. We study special functions called **isometries** which preserve the distance between points. We show translations and rotations maintain the distance between points. Reflections also are seen to be isometries. It happens that these examples are exhaustive. We are able to prove that any isometry which fixes 0 must be an orthgonal transformation. Note, orthogonal transformations are linear transformations which makes them espcially nice. Then, with a bit more effort, we show that an abitrary isometry is simply a composite of a translation and orthogonal transformation which is sometimes called a **rigid motion**.

With points settled, we move on to vectors. We see that the push forwards of isometries are orthogonal transformations on tangent vectors. We note the velocity α' is naturally transformed by the push forward for any smooth function, but, for an isometry we get derivatives $\alpha'', \alpha''', \dots$ preserved under the push forward. For an arbitrary smooth function the higher derivatives would not transform so simply, but, we don't investigate that further here[1]. The generality of this section is not really needed for following O'neill, but, I take a couple pages to share the theory of Frenet curves as given in Kühnel. This natural extension to n-dimensions brings us to study curvatures $\kappa_1, \dots, \kappa_{n-2}, \kappa_{n-1}$ where the last curvature serves as the torsion. The Frenet Serret equations in $\mathbb{R}^n$ are very similar and we sketch solutions for the constant curvature case.

In the last section we return to work on explicit results for frames and curves in the exclusively three dimensional case. We show a pair of frames at distinct points can be connected by a unique isometry. Moreover, the formula for this isometry is simply formulated in terms of the attitudes of the frames involved. We also show the push forward preserves the dot and cross product. This allows us to show the Frenet frame, curvature and torsion are nearly preserved by the push forward of an isometry. The one caveat, the sign of the torsion can be reversed if the isometry is formed by an isometry which has an orthogonal transformation which is not a pure rotation. We conclude by

[1]see page 116 of O'neill for an example

defining **congruence** and we prove that two arclength parametrized curves are congruent if and only if they share the same curvature and torsion functions.

3.1 isometries of euclidean space

We are primarily interested in three dimensional space, but, as it requires little extra effort and it actually simplifies some proofs we adopt an n-dimensional view. We use the euclidean distance function $d(p,q) = ||q-p||$ where $||v|| = \sqrt{v \bullet v}$ in what follows:

Definition 3.1.1. *isometry in $\mathbb{R}^n$:*

We say $F : \mathbb{R}^n \to \mathbb{R}^n$ is **an isometry** of $\mathbb{R}^n$ when $d(F(p),F(q)) = d(p,q)$ for all $p,q \in \mathbb{R}^n$.

In other words, an isometry of $\mathbb{R}^n$ is a function on $\mathbb{R}^n$ which *preserves distances.* The set of all isometries naturally forms a group[2] with respect to the composition of functions.

Proposition 3.1.2. *composition of isometries is an isometry:*

If F, G are isometries of $\mathbb{R}^n$ then $F \circ G$ is an isometry is an isometry of $\mathbb{R}^n$.

Proof: suppose $F, G : \mathbb{R}^n \to \mathbb{R}^n$ such that $d(F(p),F(q)) = d(p,q)$ and $d(G(p),G(q)) = d(p,q)$ for all $p,q \in \mathbb{R}^n$ then notice $G(p), G(q) \in \mathbb{R}^n$ hence:

$$d((F \circ G)(p),(F \circ G)(q)) = d((F(G(p)),F(G(q))) = d(G(p),G(q)) = d(p,q).$$

for all $p,q \in \mathbb{R}^n$. Thus $F \circ G$ is an isometry of $\mathbb{R}^n$ $\square$

There are three examples of isometries which should be familar to the reader:

Example 3.1.3. *T is a* **translation** *if there exists $a \in \mathbb{R}^n$ for which $T(x) = x + a$ for all $x \in \mathbb{R}^n$.*

$$d(T(p),T(q)) = ||T(q) - T(p)|| = ||(q+a) - (p+a)|| = ||q-p|| = d(p,q).$$

It is also interesting to note the *group theoretic* properties of the set of all translations on $\mathbb{R}^n$. Suppose we denote[3] the set of all translations by $\mathcal{T}$ then if $T, S \in \mathcal{T}$ then $T \circ S \in \mathcal{T}$ and if $T(x) = x + a$ then $T^{-1}(x) = x - a$ and we see $T^{-1} \in \mathcal{T}$. Finally, $Id(x) = x + 0$ thus identify $Id \in \mathcal{T}$ and we see $\mathcal{T}$ forms a group with respect to composition of functions. The action of translations on $\mathbb{R}^n$ is *transitive*. In particular, if $p,q \in \mathbb{R}^n$ then there exists $T \in \mathcal{T}$ for which $T(p) = q$. To see this is true just observe $T(x) = x + a$ with $a = q - p$ has $T(p) = q$ and clearly $T \in \mathcal{T}$.

Example 3.1.4. *R is a* **orthogonal transformation** *if R is a linear transformation for which $[R]^T[R] = I$ where $[R]$ denotes the standard matrix of $R : \mathbb{R}^n \to \mathbb{R}^n$. A typical example in $\mathbb{R}^3$ is given by the rotation about the z-axis:*

$$[R] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

which has $det([R]) = 1$ or the reflection $S(x) = -x$ which has $det([S]) = -1$.

[2] the proof of closure under inverse and existence of identity are implicitly covered later in this section.

[3] I actually am unaware of any standard notation

Generally, R is a rotation if it is an orthogonal transformation for which $det[R] = 1$. We define $O(n) = \{M \in \mathbb{R}^{n\times n} \mid M^TM = I\}$. Observe, $M \in O(n)$ has

$$det(M^TM) = det(I) \quad \Rightarrow \quad det(M^T)det(M) = det(M)^2 = 1 \quad \Rightarrow \quad det(M) = \pm 1.$$

The set of **orthogonal matrices** *$O(n)$ has both rotations ($det(M) = 1$) and reflections ($det(M) = -1$). The set of all rotation matrices is called the* **special orthogonal group** *of matrices (denoted $SO(n)$). Both $SO(n)$ and $O(n)$ form groups with respect to matrix multiplication. Just as in the case of translations, we can prove the composition of orthogonal transformations is an orthogonal transformations and each orthogonal transformations has an inverse transformation which is also a orthogonal transformation. Indeed, $[Id] = I$ and $I^TI = I$ hence the identity is an orthogonal transformation. Orthogonal transformations form a group with respect to function composition. However, orthogonal transformations do* **not** *act transitively on $\mathbb{R}^n$. Indeed, if R is an orthogonal transformation then $||R(p)|| = ||p||$ hence R maps spheres to spheres. We can prove the group of orthogonal transformations acts transitively on a sphere centered at the origin in $\mathbb{R}^n$. In the next section we'll see that the push-forward of an orthogonal transformation sends a frame to a frame and the push-forward of a rotation sends a positively oriented frame to a positively oriented frame*[4]

To prove orthogonal transformations are isometries, consider:

$$\begin{aligned}
d(R(p), R(q))^2 &= ||R(q) - R(p)||^2 && \textit{definition of } d \\
&= ||R(q-p)||^2 && \textit{linearity of } R \\
&= ||[R](q-p)||^2 && \textit{definition of } [R] \\
&= ([R](q-p))^T[R](q-p) && \textit{identity } ||v||^2 = v^Tv. \\
&= (q-p)^T[R]^T[R](q-p) && \textit{socks-shoes } (AB)^T = B^TA^T \\
&= (q-p)^T(q-p) && \textit{defn. of orthogonal trans.} \\
&= ||q-p||^2 && \textit{identity } ||v||^2 = v^Tv. \\
&= d(p,q)^2 && \textit{definition of distance.}
\end{aligned}$$

But, as distance is non-negative, it follows $d(R(p), R(q)) = d(p, q)$ for all $p, q \in \mathbb{R}^n$ hence every orthogonal transformation is an isometry.

We know that there are isometries which are not orthogonal transformations since translations are not generally orthogonal transformations. In fact, translations are not generally **linear transformations**. Consider $T : \mathbb{R}^n \to \mathbb{R}^n$ with $T(x) = x + a$. Observe,

$$T(0) = 0 + a = a$$

only for $a = 0$ do we obtain $T(0) = 0$ which is essential for a linear transformation. Indeed, $a = 0$ is the one special case that a translation is an orthogonal transformation. It turns out, an isometry F for which $F(0) = 0$ is an orthogonal transformation. In summary:

Theorem 3.1.5. *an isometry fixing 0 is an orthogonal transformation.*

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F(0) = 0$ then F is an isometry iff F is an orthogonal transformation.

[4] a frame is positively oriented if it has an attitude matrix which is SO(n)-valued

Proof: the converse direction of the theorem was already shown in Example 3.1.4 as, by definition, orthogonal transformation F is linear hence $F(0) = 0$.

Let us suppose F is an isometry for which $F(0) = 0$. Notice $d(F(p), F(0)) = d(p, 0)$ implies $d(F(p), 0) = d(p, 0)$ which gives $||F(p)|| = ||p||$ for all $p \in \mathbb{R}^n$. Since $||F(p)|| = ||p||$ it follows $p \neq 0$ implies $F(p) \neq 0$. Therefore, F is not identically zero. Furthermore, we can show the image of F includes an orthogonal set of n nontrivial vectors. Consider the standard basis of $\mathbb{R}^n$, $(e_i)^j = \delta_{ij}$. Observe $F(e_i) \bullet F(e_j) = e_i \bullet e_j = \delta_{ij}$. Thus $\{F(e_1), F(e_2), \dots, F(e_n)\}$ is an orthogonal basis for $\mathbb{R}^n$.

Let $p, q \in \mathbb{R}^n$ and note:

$$\begin{aligned} d(F(p), F(q))^2 &= ||F(q) - F(p)||^2 \\ &= (F(q) - F(p)) \bullet (F(q) - F(p)) \\ &= F(q) \bullet F(q) - 2F(p) \bullet F(q) + F(p) \bullet F(p) \\ &= ||F(q)||^2 - 2F(p) \bullet F(q) + ||F(p)||^2 \quad \star \end{aligned}$$

likewise,

$$d(p, q)^2 = ||q - p||^2 = ||q||^2 - 2p \bullet q + ||p||^2 \quad \star^2 .$$

Therefore, as $d(F(p), F(q)) = d(p, q)$, we compare $\star$ and $\star^2$ and use $||F(p)|| = ||p||$ and $||F(q)|| = ||q||$ to derive $F(p) \bullet F(q) = p \bullet q$ for all $p, q \in \mathbb{R}^n$. We seek to show F is a linear transformation: suppose $x, y, z \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$F(cx + y) \bullet F(z) = (cx + y) \bullet z = cx \bullet z + y \bullet z = cF(x) \bullet F(z) + F(y) \bullet F(z)$$

thus

$$F(cx + y) \bullet F(z) - cF(x) \bullet F(z) - F(y) \bullet F(z) = 0.$$

By algebra of the dot product,

$$\big[F(cx + y) - cF(x) - F(y)\big] \bullet F(z) = 0.$$

But, if we take $z = e_i$ for $i = 1, 2, \dots, n$ then the equation above shows the vector $F(cx + y) - cF(x) - F(y)$ is zero in the $F(e_i)$-direction for $i = 1, 2, \dots, n$ hence $F(cx + y) - cF(x) - F(y) = 0$. Therefore, $F(cx + y) = cF(x) + F(y)$ for all $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Finally, as F is linear, the linear algebra gives a matrix R for which $F(x) = Rx$ for all $x \in \mathbb{R}^n$ and it follows

$$y^T I z = y \bullet z = F(y) \bullet F(z) = (Ry) \bullet (Rz) = y^T R^T R z$$

hence $R^T R = I$ and we conclude F is an orthogonal transformation. $\square$

If we drop the requirement that $F(0) = 0$ for an isometry then the possibilities are still quite limited. It turns out that every isometry of $\mathbb{R}^n$ is simply a composition of a translation and an orthogonal transformation.

Theorem 3.1.6. *an isometries are generated by translations and orthgonal transformations*

Every isometry $F : \mathbb{R}^n \to \mathbb{R}^n$ can be written uniquely as $F = T \circ R$ where T is a translation and R is an orthogonal transformation. That is, there exists $M \in \mathrm{O}(n)$ and $a \in \mathbb{R}^n$ such that $F(x) = Mx + a$ for each $x \in \mathbb{R}^n$.

Proof: first, we should note that if $F = T \circ R$ for R an orthogonal transformation and $T(x) = x + a$ then, as we have already shown translations and rotations are isometries, Proposition 3.1.2 shows F is an isometry.

Conversely, suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry. Let $G(x) = F(x) - F(0)$. Notice G is an isometry since it is the composition of F and the translation by $-F(0)$. Furthermore, note $G(0) = F(0) - F(0) = 0$ thus G is an orthogonal transformation by Theorem 3.1.5. Hence $F(x) = G(x) + F(0)$ and we identify $F = T \circ G$ where G is an orthogonal transformation and $T(x) = x + F(0)$. It follows, $F(x) = Mx + a$ for $M = [G] \in \mathrm{O}(n)$ and $a = F(0)$. To prove uniqueness, suppose $F(x) = Nx + b$ for some $N \in \mathrm{O}(n)$ and $b \in \mathbb{R}^n$. Consider, $Mx + a = Nx + b$ for all $x \in \mathbb{R}^n$. Set $x = 0$ to see $a = b$. Likewise, setting $x = e_j$ we obtain the $Me_j = Ne_j$ hence $col_j(M) = col_j(N)$ for $j = 1, \dots, n$ hence $M = N$. $\square$

Finally, I ought to mention, the set of all isometries for $\mathbb{R}^n$ forms a group. This is known as the group of **rigid motions** as it includes only those transformations which maintain the shape of rigid objects[5]. These **euclidean motions** take squares to squares, triangles to triangles, and so forth. The preservation of the dot-product between vectors means the angle between vectors is maintain in the image of a rigid motion. Moreover, the lengths of vectors are also maintained. A rigid motion just reflects and or rotates then translates a given shape. We consider how the push-forward of an isometry treats vectors in the next section.

3.2 how isometries act on vectors

Let us begin by examining how the velocity of a curve is naturally transformed under the push foward of an arbitrary smooth map on $\mathbb{R}^n$. If $\alpha : I \to \mathbb{R}^n$ is a smooth parametrized curve then we defined the velocity by $\alpha'(t) = \sum_{i=1}^{n} \frac{d\alpha^i}{dt} \frac{\partial}{\partial x^i}\Big|_{\alpha(t)}$. Generally, if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth mapping then $F \circ \alpha$ defines a curve in the image of F and we have the **chain rule**:

$$(F \circ \alpha)'(t) = \sum_{i=1}^{n} \frac{d(F \circ \alpha)^i}{dt} \frac{\partial}{\partial x^i}\Big|_{F(\alpha(t))} = \sum_{i,j=1}^{n} \frac{\partial F^j}{\partial x^i} \frac{d\alpha^i}{dt} \frac{\partial}{\partial x^i}\Big|_{\alpha(t)} = d_{\alpha(t)}F(\alpha'(t))$$

where we identified push forward of $\alpha'(t)$ by $d_{\alpha(t)}F$ in the last equality. In summary:

Theorem 3.2.1. *the push foward of the velocity is the velocity of the image curve.*

> If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map and $\alpha : I \to \mathbb{R}^n$ is a parametrized smooth curve then
>
> $$F_*(\alpha'(t)) = (F \circ \alpha)'(t)$$

We should allow a bit of notation here as to avoid too much writing. The tangent space $T_p\mathbb{R}^n$ has the natural basis $\partial_i|_p$ for $i = 1, 2, \dots, n$. Likewise $T_{F(p)}\mathbb{R}^n$ has the natural basis $\partial_i|_{F(p)}$ for $i = 1, 2, \dots, n$. The vector X with components X^i is transformed to the vector $F_*(X)$ with components $\sum_j \frac{\partial F^j}{\partial x^i} X^j$. Identify this is just the usual matrix-colum product and consequently, with the identifications just set out understood, the Theorem above is written:

$$F_*(X) = J_F X. \tag{3.1}$$

[5] there is a matrix version of this group of euclidean motions which I will probably show in homework

Where $(J_F)_{ij} = \frac{\partial F^j}{\partial x^i}$ in the usual linear algebra index notation.

If F is an isometry of $\mathbb{R}^n$ then $F(x) = Rx + a$ for all $x \in \mathbb{R}^n$ for some $R \in \mathrm{O}(n)$ and $a \in \mathbb{R}^n$. Note,

$$F^i(x) = \sum_{j=1}^n R^i_j x^j + a^i \quad \Rightarrow \quad \frac{\partial F^i}{\partial x^k} = \sum_{j=1}^n \frac{\partial}{\partial x^k}\left[R^i_j x^j\right] + \frac{\partial}{\partial x^k}\left[a^i\right] = \sum_{j=1}^n R^i_j \frac{\partial x^j}{\partial x^k} = R^i_j.$$

The calculation above shows the Jacobian matrix of an isometry is simply the orthogonal matrix for the orthogonal transformation of the isometry. Thus, in view of Equation 3.1 we find:

$$F_*(v) = Rv \tag{3.2}$$

where to be more carefull, $F_*(p, v) = (F(p), Rv)$. Let me cease this ritual of notation and say something interesting: Theorem 3.2.1 applied to an isometry gives us the result that the push forward of a k-jet[6] is the k-jet of the image curve under the isometry.

Recall we defined α'' for a parametrized curve in $\mathbb{R}^3$. We define the k-th derivative of α in $\mathbb{R}^n$ in the same fashion:

$$\alpha^{(k)}(t) = \sum_{i=1}^n \frac{d^k\alpha^i}{dt^k}\frac{\partial}{\partial x^i}\bigg|_{\alpha(t)} = \left(\alpha(t), \left\langle \frac{d^k\alpha^1}{dt^k}, \dots, \frac{d^k\alpha^n}{dt^k}\right\rangle\right).$$

Thus, $\alpha'' = (\alpha')'$ and generally $\alpha^{(k)} = (\alpha^{(k-1)})'$ for $k \in \mathbb{N}$ where we use Defn. 2.3.3 to differentiate the velocity vector fields of velocity, acceleration and so forth along α.

Theorem 3.2.2. *natural transfer of k-jet by isometry of $\mathbb{R}^n$.*

Suppose an isometry $F : \mathbb{R}^n \to \mathbb{R}^n$ is written as $F(x) = Rx + a$ for $a \in \mathbb{R}^n$ and $R \in \mathrm{O}(n)$. Also, suppose $\alpha : I \to \mathbb{R}^n$ is a smooth parametrized curve. Then: for $k \in \mathbb{N}$,

$$F_*(\alpha') = (F\circ\alpha)', \quad F_*(\alpha'') = (F\circ\alpha)'', \quad \dots, \quad F_*(\alpha^{(k)}) = (F\circ\alpha)^{(k)}.$$

Moreover, the magnitude of $\alpha^{(i)}$ angle between $\alpha^{(i)}$ and $\alpha^{(j)}$ is preserved under F_*.

$$||F_*(\alpha^{(i)}(t))|| = ||\alpha^{(i)}(t)|| \quad \& \quad F_*(\alpha^{(i)}(t)) \bullet F_*(\alpha^{(j)}(t)) = \alpha^{(i)}(t) \bullet \alpha^{(j)}(t)$$

for each $t \in I$ and $i, j = 1, 2, \dots, n$.

Proof: in Definition 2.3.6 Use Equation 3.2 and set $v = \alpha^{(k)}(t)$

$$F_*(\alpha^{(k)}(t)) = R\alpha^{(k)}(t)$$

for $k \in \mathbb{N}$. On the other hand the image curve $F \circ \alpha$ has:

$$(F\circ\alpha)(t) = R\alpha(t) + a \quad \Rightarrow \quad (F\circ\alpha)^{(k)}(t) = R\alpha^{(k)}(t) = F_*(\alpha^{(k)}(t)).$$

Omitting the t-dependence,

$$F_*(\alpha^{(k)}) = R\alpha^{(k)}$$

[6]see page 12 of Kühnel for a related concept of the k-th order contact of two curves. The k-jet of a curve can be identified with an equivalence class of curves whose initial k derivatives align at the given point

Hence angles between derivatives and their magnitudes of α are preserved under the push-forward of an isometry since orthogonal matrices preserve the dot product and hence the norm and angles. $\square$

Specialize to $n = 3$ for a moment. Given a nonlinear regular curve α we defined T, N, B along α. Moreover, T, N, B are all derived from dot and cross products of α' and α'' hence the curvature and torsion of α and $F \circ \alpha$ will match perfectly provided we preserve the cross-product[7]. It turns out that forces us to require the isometry has $\det(R) = 1$.

We should also pause to consider the significance of Theorem 3.2.2 to Newtonian Mechanics. The central equation of mechanics is $F_{net} = ma$. Of course, this equation is made with respect to a particular coordinate system. If we change to another coordinate system which is related to the first by an isometry then the acceleration naturally transforms. Of course, the story is a bit more complicated since the coordinate systems in physics also can *move*. We could think about the T, N, B frame moving with the curve. Indeed, many of the annoyning gifs on my webpage are based on this idea. But, the isometries we've studied in this chapter have no time dependence. That said, allowing the coordinate systems to move just introduces an additional constant velocity motion in addition to the isometries we primarily study here. If you'd like to read a somewhat formal treatment of Newtonian Mechanics then you might enjoy Chapter 6 of my notes from Math 430 which I taught as a graduate student at NCSU. I describe something called a *Euclidean Structure* which gives a natural mathematical setting to describe the concept of an observer and a moving frame of reference. Then in Chapter 7 of those notes I describe a *Minkowski Structure* which does the same for Special Relativity[8]. I don't think I'll cover it, but I should mention Kühnel treats curves in Minkowski space and the low-dimensional geometry of curves and surfaces in manifolds with Lorentzian geometry is still quite active.

3.2.1 Frenet curves in $\mathbb{R}^n$

Let me pause from our general development to introduce the theory of Frenet curves from Chapter 2 of Wolfgang Kühnel's *Differential Geometry : Curves-Surfaces-Manifolds*. Here we find a generalization of the T, N, B construction for n-dimensional space and the results we have proved thus far for push fowards of isometries will go to prove that a Frenet curve and its isometric image have all same curvatures and the same torsion if the isometry is comprised of a rotation and translation.

Definition 3.2.3. *frame in $\mathbb{R}^n$:*

If $E_1, \dots, E_n \in \mathfrak{X}(\mathbb{R}^n)$ are orthonormal vector fields then we say $\{E_1, \dots, E_n\}$ is a **frame** of $\mathbb{R}^n$. If $E_i = \sum_j A_{ij} E_j$ then A is the **attitude matrix** of the frame. Frames for which $\det(A) = 1$ are called **positively oriented** frames and frames with $\det(A) = -1$ are said to be **negatively oriented**.

In $\mathbb{R}^n$ we use determinant to play the role the cross-product played in $\mathbb{R}^3$. If we have $n-1$ orthonormal vectors $E_1, \dots, E_{n-1}$ then the vector $E_n = \sum_j (E_n)^j U_j$ will be defined by

$$(E_n)^j = \det \left[E_1 | \dots | E_{n-1} | U_j \right]. \tag{3.3}$$

In the construction of the Frenet frame described below the first $(n-1)$ vectors in the frame are generated by the Gram-Schmidt algorithm applied the linearly independent derivatives and higher

[7] I think this is quite plausible, but we will prove it later, see Theorem 3.3.4

[8] I should be clear, these notes closely parallel a presentation of Math 430 I enjoyed as an undergraduate from R.O. Fulp and certainly the main mathematical ideas are his and the mistakes are most likely mine.

derivatives of the curve to $(n-1)$-th order. The remaining vector can be generated from the **generalized cross product** I described in Equation 3.3.

Definition 3.2.4. *Frenet curve in $\mathbb{R}^n$:*

Suppose $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^n$ is a regular arclength parametrized curve. We say α is a **Frenet curve** if $\alpha', \alpha'', \dots, \alpha^{(n-1)}$ are linearly independent. Furthermore, a set of n-vector fields $E_1, E_2, \dots, E_n$ is a **Frenet frame** of α if the following three conditions are met:

(i.) $E_1, E_2, \dots, E_n$ are orthonormal and positively oriented,

(ii.) for each $k = 1, 2, \dots, n-1$ we have $\text{span}\{E_1, \dots, E_k\} = \text{span}\{\alpha', \dots, \alpha^{(k)}\}$

(iii.) $\alpha^{(k)} \bullet E_k > 0$ for $k = 1, \dots, n-1$.

The condition of linear independence in $n = 3$ simply amounts to the assumption $\alpha'' \neq 0$. In other words, a regular non-linear arclength-parametrized curve is a Frenet curve and you can verify that T, N, B frame is a Frenet frame. The Frenet Serret equations also have a generalization to Frenet curves in $\mathbb{R}^n$. In particular:

Theorem 3.2.5. *Frenet Serret equations in $\mathbb{R}^n$.*

Let α be a Frenet curve and $E_1, E_2, \dots, E_n$ a Frenet frame then there exist non-negative **curvature functions** $\kappa_i = E_i' \bullet E_{i+1}$ for $i = 1, 2, \dots, n-2$ and **torsion** $\kappa_{n-1} = E_{n-1}' \bullet E_n$ for which the following differential equations hold true:

$$\begin{bmatrix} E_1' \\ E_2' \\ \vdots \\ E_{n-1}' \\ E_n' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & \cdots & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & \ddots & \ddots & \vdots \\ \vdots & -\kappa_2 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \kappa_{n-1} \\ 0 & 0 & \cdots & \cdots & -\kappa_{n-1} & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_{n-1} \\ E_n \end{bmatrix}$$

Proof: is found on page 27 of Kühnel. It is similar to the proof we gave for the $n = 3$ case. $\square$

Notice, by Theorem 3.2.2, if we are given an isometry F then we may push a Frenet frame $E_1, \dots, E_n$ to α at $\alpha(s_o)$ to another Frenet frame $F_*(E_1), \dots, F_*(E_n)$ at $F(\alpha(s_o))$ for the curve $F \circ \alpha$. Notice

$$\det(F_*(E_1)|\cdots|F_*(E_n)) = \det(RE_1|\cdots|RE_n) = \det(R[E_1|\cdots|E_n]) = \det(R)\det(E_1|\cdots|E_n) \tag{3.4}$$

thus we need $\det(R) = 1$ to give the $F_*(E_i)$ frame a positive orientation. It turns out that $\det(R) = -1$ causes the torsion to reverse sign. However, the lower curvatures are preserved under the push forward of an arbitrary isometry of $\mathbb{R}^n$ since they are defined just in terms of dot-products of the pushed-forward frame which all agree with the initial frame[9]. The torsion is special in that it involves the determinant.

One interesting case is that $\kappa_{n-1} = 0$ for the curve. If the curve satisfies this vanishing torsion requirement then E_n serves as normal vector to the hyperplane in which the curve is found. This is a natural generalization of the $\tau = 0$ implies planar motion in $n = 3$. Another interesting example is seen in the lower dimensional case:

[9]See Lemma 2.14 of page 28 of Kühnel for a detailed argument.

Example 3.2.6. *Suppose we have a Frenet curve in* $\mathbb{R}^2$. *It suffices to have a regular curve; that is* $\alpha : I \to \mathbb{R}^2$ *with* $\alpha'(s) \neq 0$ *for all* $s \in I$. *If* E_1, E_2 *is defined by:*

$$E_1(s) = \alpha'(s) = a(s)U_1 + b(s)U_2 \quad \& \quad E_2(s) = -b(s)U_1 + a(s)U_2$$

where the unit-speed assumption means the functions a, b *must satisfy* $a^2 + b^2 = 1$. *We define curvature* $\kappa = ||E_1'||$ *which gives:*

$$\kappa = \sqrt{(a')^2 + (b')^2}$$

For example, $\alpha(s) = (R\cos(s/R), R\sin(s/R))$ *gives* $\alpha'(s) = -\sin(s/R)U_1 + \cos(s/R)U_2 = E_1$ *thus*

$$E_1' = \tfrac{-\cos(s/R)}{R}U_1 - \tfrac{\sin(s/R)}{R}U_2 \quad \Rightarrow \kappa = \tfrac{1}{R}.$$

The reverse question is also very interesting. In particular, given a set of curvature functions and a torsion function when can we find a Frenet curve which takes the given functions as its curvature and torsion?

Theorem 3.2.7. *on solving Frenet Serret equations in* $\mathbb{R}^n$.

Suppose we are given $(n-1)$ smooth functions $\kappa_i : (a,b) \to \mathbb{R}$ for which $\kappa_i(s) \geq 0$ for $s \in (a,b)$ and $i = 1, 2, \dots, n-2$. Also, suppose a point q_o and a frame $v_1, \dots v_n \in T_{q_o}\mathbb{R}^n$ is given. Let $s_o \in (a,b)$. There exists a unique Frenet curve α for which $\alpha(s_0) = q_o$ and the curve has Frenet frame $E_1, \dots, E_n$ for which $E_i(s_o) = v_i$ for $i = 1, \dots, n$ and $\kappa_i = E_i' \bullet E_{i+1}$ for $i = 1, \dots, n-1$.

Proof: a slightly improved version of the above theorem as well as the proof is found on page 28 of of Kühnel. Ultimately this theorem, like so many others, rests on the existence and uniqueness theorem for solutions of systems of ordinary differential equations. $\square$

We found in Example 3.2.6 that a circle has constant curvature which is reciprocal to its radius. The example which follows investigates the reverse question:

Example 3.2.8. *Find the arclength parametrized curve in* $\mathbb{R}^2$ *for which* $\kappa(s) = 1/R$ *for all* s. *We seek to solve:*

$$E_1' = \kappa E_2, \qquad E_2' = -\kappa E_1$$

Differentiate the first equation and subsitute the second equation to obtain:

$$E_1'' = -\kappa^2 E_1$$

This gives[10] $E_1 = \cos(\kappa s)C_1 + \sin(\kappa s)C_2$ *for constant vectors* C_1, C_2. *Next, by* $E_2 = \frac{1}{\kappa}E_1'$ *we derive:*

$$E_2 = -\sin(\kappa s)C_1 + \cos(\kappa s)C_2$$

Finally, the curve itself is found from integrating $\alpha'(s) = E_1(s)$:

$$\alpha(s) = \alpha_o + \frac{1}{\kappa}\sin(\kappa s)C_1 - \frac{1}{\kappa}\cos(\kappa s)C_2.$$

Of course, the constant vectors C_1, C_2 *must be specified such that* $||E_1|| = 1$ *and* $||E_2|| = 1$. *Observe,*

$$||\alpha(s) - \alpha_o|| = \frac{1}{\kappa}||\sin(\kappa s)C_1 - \cos(\kappa s)C_2|| = \frac{1}{\kappa}||-E_2|| = R.$$

Just as we hoped, this is a circle of radius $R = 1/\kappa$ *centered at* α_o.

[10] in terms of the usual ODEs course, think of $E_1 = yU_1 + zU_2$ thus $E_1'' + \kappa^2 E_1 = 0$ yields $y'' + \kappa^2 y = 0$ and $z'' + \kappa^2 z = 0$. Then the usual techniques yield $y = C_{11}\cos\kappa s + C_{12}\sin\kappa s$ and $z = C_{21}\cos\kappa s + C_{22}\sin\kappa s$ hence combining those we obtain my claimed vector solution with $C_1 = (C_{11}, C_{12})$ and $C_2 = (C_{21}, C_{22})$

If you've studied constant coefficient ordinary differential equations then it should be clear we can solve the constant curvature case in arbitrary dimension. In $n = 2$ we obtain a circle. In $n = 3$ we could derive a helix generally. In $n = 4$ let's see what happens:

Example 3.2.9. *Suppose κ_1, κ_2, τ are constants. Solve the Frenet Serret equations*[11] *for the frame E_1, E_2, E_3, E_4 which gives constant curvatures κ_1, κ_2, τ:*

$$\begin{bmatrix} E_1' \\ E_2' \\ E_3' \\ E_4' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & -\kappa_2 & 0 & \tau \\ 0 & 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix}$$

Direct calculation as in the $n = 2$ example is fairly challenging here. Instead, look at the system above as $E' = ME$ and use the matrix exponential $\exp(sM)$ to solve the system. We need only find eigenvalues and eigenvectors to extract solutions from the matrix exponential. Consider,

$$det(xI - M) = det \begin{bmatrix} x & -\kappa_1 & 0 & 0 \\ \kappa_1 & x & -\kappa_2 & 0 \\ 0 & \kappa_2 & x & -\tau \\ 0 & 0 & \tau & x \end{bmatrix} = x\,det \begin{bmatrix} x & -\kappa_2 & 0 \\ \kappa_2 & x & -\tau \\ 0 & \tau & x \end{bmatrix} + \kappa_1 det \begin{bmatrix} \kappa_1 & -\kappa_2 & 0 \\ 0 & x & -\tau \\ 0 & \tau & x \end{bmatrix}.$$

Then,

$$det \begin{bmatrix} x & -\kappa_2 & 0 \\ \kappa_2 & x & -\tau \\ 0 & \tau & x \end{bmatrix} = x[x^2 + \tau^2 + \kappa_2^2] \quad \& \quad det \begin{bmatrix} \kappa_1 & -\kappa_2 & 0 \\ 0 & x & -\tau \\ 0 & \tau & x \end{bmatrix} = \kappa_1(x^2 + \tau^2).$$

Thus,

$$det(xI - M) = x^2[x^2 + \tau^2 + \kappa_2^2] + \kappa_1^2(x^2 + \tau^2) = x^4 + (\tau^2 + \kappa_2^2 + \kappa_1^2)x^2 + \kappa_1^2\tau^2.$$

The reader will trust that the formulas for the solutions of the above equation is cumbersome. Let us agree the solution comes in a pair of complex zeros, $x = \pm i\gamma$ and $x = \pm i\beta$. Notice $trace(M) = 0$ is consistent with this claim and we also expect $\gamma^2\beta^2 = \kappa_1\tau^2 = det(M)$. Details aside, the solution has the form: for constant vectors C_1, C_2, C_3, C_4,

$$E_1 = \cos(\gamma s)\, C_1 + \sin(\gamma s)\, C_2 + \cos(\beta s)\, C_3 + \sin(\beta s)\, C_4$$

from which we can derive $\alpha(s)$ by integration and E_2, E_3, E_4 by the Frenet Serret equations. See page 32 of Kühnel for more on how the curvatures are explicitly related to the coefficients of the solution. Apparently, these can give curves which wind around some torus in four dimensional space.

I should mention, the remainder of Chapter 2 in Kühnel discusses other topics I will not probably touch in these notes. Particularly interesting is the treatment of curves in three dimensional Minkowski space. Some of the material on total curvature we will return to much later as we study the Gauss Bonnet Theorem.

[11] This is an interesting differential equation. Technically, it has 16 ODEs, but, they are arranged such that we can solve it as if the frame fields were just ordinary real-valued variables.

3.3 on frames and congruence in three dimensions

Much of what is done in this section for $\mathbb{R}^n$, but, I resist the temptation and simply work in $\mathbb{R}^3$.

We saw after Example 3.1.3 that translations act transitively on $\mathbb{R}^3$. Something similar is true for isometries and frames in $\mathbb{R}^3$; suppose we have a pair of frames E_1, E_2, E_3 and F_1, F_2, F_3 at p. We can find an orthogonal transformation Φ which transfers E to F. We seek Φ for which

$$\Phi_*(E_1) = F_1, \quad \Phi_*(E_2) = F_2, \quad \Phi_*(E_3) = F_3.$$

If the matrix of Φ_* is R then

$$R[E_1] = [F_1], \quad R[E_2] = [F_2], \quad R[E_3] = [F_3]$$

where $E_i = A_{i1}U_1 + A_{i2}U_2 + A_{i3}U_3$ for $i = 1, 2, 3$ implies $[E_i] = [A_{i1}, A_{i2}, A_{i3}]^T$. Likewise, $F_i = B_{i1}U_1 + B_{i2}U_2 + B_{i3}U_3$. Concatenate the three equations above into a single equation to obtain:

$$RA^T = B^T \quad \Rightarrow \quad R = B^T A.$$

Since the attitude matrices A, B exist for a given pair of frames at p it follows that $\Phi(x) = B^T Ax$ defines the desired isometry which pushes the E-frame to the F-frame. With this calculation in hand, the theorem below is easy to prove:

Theorem 3.3.1. *transfer of frame by isometry.*

Let $E_1, E_2, E_3 \in T_p\mathbb{R}^3$ be a frame at p with attitude A and $F_1, F_2, F_3 \in T_q\mathbb{R}^3$ be a frame at q with attitude B then there exists an isometry $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ for which $\Phi(p) = q$ and

$$\Phi_*(E_1) = F_1, \quad \Phi_*(E_2) = F_2, \quad \Phi_*(E_3) = F_3.$$

Proof: let $R = B^T A$ and define $\Phi(x) = Rx + q - Rp$ for all $x \in \mathbb{R}^3$. Observe $\Phi(p) = Rp + q - Rp = q$ as desired. Furthermore, as $A^T A = I$ we find $R = B^T A$ implies $RA^T = B^T$. But, this shows the matrix R maps each row of A to the corresponding row of B. Hence the push-forward with matrix R maps vectors with coordinates formed by rows of A to new vectors with coordinates formed by rows of B. But, E_i has coordinates in $[\text{row}_i(A)]^T$ and F_i has coordinates in $[\text{row}_i(B)]^T$ thus

$$\Phi_*(E_i) = F_i$$

where E_i is at p and F_i is at $\Phi(p) = q$. $\square$

Let me unpack the proof above in gory detail:

$$\begin{aligned}
\Phi_*(E_i) &= \Phi_*\big(\sum_j A_{ij} U_j(p) \big) && \text{defn. of attitude } A \\
&= \sum_j A_{ij} \Phi_*\big(U_j \big) && \text{property of push forward} \\
&= \sum_j A_{ij} \sum_{k,l} \frac{\partial \Phi^k}{\partial x^l} (U_j)^l U_k(\Phi(p)) && \text{defn. of push forward} \\
&= \sum_{j,k} A_{ij} \frac{\partial \Phi^k}{\partial x^j} U_k(q) && (U_j)^l = U_j[x^l] = \delta_{jl} \\
&= \sum_{j,k} A_{ij} R^k{}_j U_k(q) && \text{took } \partial_j \text{ of } \Phi(x) = Rx + q - Rp \\
&= \sum_{j,k} R^k{}_j (A^T)_{ji} U_k(q) && \text{defn. of transpose} \\
&= \sum_k (RA^T)^k{}_i U_k(q) && \text{defn. of transpose} \\
&= \sum_k (RA^T)^T_{ik} U_k(q) && \text{defn. of transpose} \\
&= \sum_k B_{ik} U_k(q) && \text{as } (RA^T)^T = (B^T)^T = B. \\
&= F_i && \text{defn. of the attitude } B
\end{aligned}$$

The nice thing about the theorem is we can calculate the isometry simply by taking the appropriate product of attitude matrices.

Example 3.3.2. *Consider the frame and attitude matrix at $p = (1, 1, 1)$ given below:*

$$\begin{array}{l} E_1 = \frac{1}{\sqrt{2}} U_1 + \frac{1}{\sqrt{2}} U_2 \\ E_2 = \frac{1}{\sqrt{2}} U_1 - \frac{1}{\sqrt{2}} U_2 \\ E_3 = U_3 \end{array} \quad \Rightarrow \quad A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Likewise at $q = (-1, -2, -3)$ another frame and corresponding attitude are given:

$$\begin{array}{l} E_1 = \frac{1}{\sqrt{3}} \left(U_1 + U_2 + U_3 \right), \\ E_2 = \frac{1}{\sqrt{2}} \left(U_1 - U_3 \right), \\ E_3 = \frac{1}{\sqrt{6}} \left(U_1 - 2U_2 + U_3 \right) \end{array} \quad \Rightarrow \quad B = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Calculate $R = B^T A$,

$$R = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c|c} \frac{1}{\sqrt{6}} + \frac{1}{2} & \frac{1}{\sqrt{6}} - \frac{1}{2} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} - \frac{1}{2} & \frac{1}{\sqrt{6}} + \frac{1}{2} & \frac{1}{\sqrt{6}} \end{array} \right]$$

Consequently,

$$Rp = \left[\begin{array}{c|c|c} \frac{1}{\sqrt{6}} + \frac{1}{2} & \frac{1}{\sqrt{6}} - \frac{1}{2} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} - \frac{1}{2} & \frac{1}{\sqrt{6}} + \frac{1}{2} & \frac{1}{\sqrt{6}} \end{array} \right] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{6}} \\ 0 \\ \frac{3}{\sqrt{6}} \end{bmatrix}$$

Thus, using $\Phi(x) = Rx + q - Rp$ we have:

$$\Phi(x,y,z) = \left[\begin{array}{c|c|c} \frac{1}{\sqrt{6}} + \frac{1}{2} & \frac{1}{\sqrt{6}} - \frac{1}{2} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} - \frac{1}{2} & \frac{1}{\sqrt{6}} + \frac{1}{2} & \frac{1}{\sqrt{6}} \end{array}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 + \frac{3}{\sqrt{6}} \\ 2 \\ 3 + \frac{3}{\sqrt{6}} \end{bmatrix}.$$

I invite the reader to verify that Φ_* pushes the E-frame to the F-frame as claimed. It is easy to check that the product of the transpose of i-th row of A with R yields the transpose of the i-th row of B hence $\Phi_*(E_i) = F_i$ for $i = 1, 2, 3$.

Let us return to the discussion we initiated with Equation 3.4.

Theorem 3.3.3. *isometries and push forward of vector products in $\mathbb{R}^3$*

Suppose $F : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry and $V, W \in \mathfrak{X}(\mathbb{R}^3)$ then

$$F_*(V) \bullet F_*(W) = V \bullet W \qquad \& \qquad F_*(V) \times F_*(W) = \det(F_*)\, F_*(V \times W).$$

Proof: following[12] Equation 3.2 suppose the orthogonal matrix for F is R,

$$\begin{aligned} F_*(V) \bullet F_*(W) &= (RV) \bullet (RW) \\ &= (RV)^T(RW) \\ &= V^T R^T R W = V^T W = V \bullet W. \end{aligned}$$

Likewise,

$$F_*(V) \times F_*(W) = (RV) \times (RW)$$

Notice, $R^T R = I$ and $F_*(U_1), F_*(U_2), F_*(U_3)$ forms a frame thus as $F_*(U_j) = RU_j$,

$$(F_*(V) \times F_*(W)) \bullet U_j = ((RV) \times (RW)) \bullet U_j = \det\big[RV|RW|U_j\big]$$

Note $R^T R = I$ allows $\big[RV|RW|RR^T U_j\big] = R\big[V|W|R^T U_j\big]$ hence

$$\det\big[RV|RW|U_j\big] = \det(R)\det\big[V|W|R^T U_j\big]$$

and as $\det(F_*) = \det(R)$ we find

$$\begin{aligned} (F_*(V) \times F_*(W)) \bullet U_j &= \det(F_*)\det\big[V|W|R^T U_j\big] \\ &= \det(F_*)(V \times W) \bullet R^T U_j \end{aligned}$$

Thus, as $(V \times W) \bullet R^T U_j = R(V \times W) \bullet U_j = F_*(V \times W) \bullet U_j$ for each $j = 1, 2, 3$ we derive:

$$F_*(V) \times F_*(W) = \det(F_*) F_*(V \times W). \qquad \square$$

The proof given above equally well applies to vector fields along some curve. You may recall from Theorem 3.2.2 we already know dot product is preserved between vector fields $\alpha', \alpha'', \dots$ along α when pushed forward to the curve $F \circ \alpha$ where F is an isometry. Also, notice $\det(F_*) = R$ forces $\det(F_*) = \pm 1$ since $R^T R = I$ implies $\det(R) = \pm 1$. It follows that only the isometries built from rotations ($\det(R) = 1$) preserve the full Frenet apparatus for a non-linear regular curve.

[12]the matrix products below are products of the coordinate vectors with respect to U_1, U_2, U_3

Theorem 3.3.4. *Frenet apparatus of curve's isometric image*

Let α be a nonlinear arclength parametrized regular curve with Frenet frame T, N, B and curvature κ and torsion τ. Suppose $F : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry and define $\bar{\alpha} = F \circ \alpha$. If $\overline{T}, \overline{N}, \overline{B}$ and $\bar{\kappa}$ and $\bar{\tau}$ form the Frenet apparatus for $\bar{\alpha}$ then

$$\overline{T} = F_*(T), \quad \overline{N} = F_*(N), \quad \overline{B} = \det(F_*)F_*(B), \quad \bar{\kappa} = \kappa, \quad \bar{\tau} = \det(F_*)\tau.$$

Proof: the unit tangents for α and $\bar{\alpha}$ are defined as $T(s) = \alpha'(s)$ and $\overline{T}(s) = \bar{\alpha}'(s) = (F \circ \alpha)'(s)$. In Theorem 3.2.2 we learned $F_*(\alpha') = (F \circ \alpha)'$ thus $\overline{T}(s) = F_*(T)$.

The Frenet normal for α and $\bar{\alpha}$ are defined by $N(s) = \frac{1}{\kappa}\alpha''$ and $\overline{N}(s) = \frac{1}{\bar{\kappa}}\bar{\alpha}''$ where $\kappa = ||\alpha''||$ and $\bar{\kappa} = ||\bar{\alpha}''||$. However, as $\bar{\alpha}'' = (F \circ \alpha)''$ and Theorem 3.2.2 provides $F_*(\alpha'') = (F \circ \alpha)''$ and $||F_*(\alpha'')|| = ||\alpha''||$ thus $||\alpha''|| = ||(F \circ \alpha)''|| = ||\bar{\alpha}''||$ we derive $\kappa = \bar{\kappa}$ and:

$$F_*(N) = F_*\left(\frac{1}{||\alpha''||}\alpha''\right) = \frac{1}{||\alpha''||}F_*(\alpha'') = \frac{1}{||(F \circ \alpha)''||}(F \circ \alpha)'' = \overline{N}.$$

The binormal of α and $\bar{\alpha}$ are defined as $B = T \times N$ and $\overline{B} = \overline{T} \times \overline{N}$. Therefore, by Theorem 3.3.3,

$$F_*(B) = F_*(T \times N) = \det(F_*)F_*(T) \times F_*(N) = \det(F_*)\overline{T} \times \overline{N} = \det(F_*)\overline{B}.$$

The torsions of α and $\bar{\alpha}$ are defined by $\tau = -B' \bullet N$ and $\bar{\tau} = -\overline{B}' \bullet \overline{N}$. Apply Theorem 3.3.3,

$$B' \bullet N = F_*(B') \bullet F_*(N)$$

I invite the reader to prove the small lemma $F_*(B') = \det(F_*)\overline{B}'$ from which we find:

$$B' \bullet N = F_*(B') \bullet F_*(N) = \det(F_*)\overline{B}' \bullet \overline{N}$$

Therefore, $\tau = \det(F_*)\bar{\tau}$. $\square$

The theorem above shows we can maintain the curvature and torsion of a curve when we transform the curve by rotation and translation.

Example 3.3.5. *Suppose* $\alpha(s) = (R\cos(s/\sqrt{R^2+m^2}), R\sin(s/\sqrt{R^2+m^2}), ms/\sqrt{R^2+m^2})$. *Let* $\theta = s/\sqrt{R^2+m^2}$. *Consider* $F(x,y,z) = (y,x,z)$. *This is a simple isometry which has* $\det(F_*) = -1$. *Consider*

$$\bar{\alpha}(s) = (F \circ \alpha)(s) = (R\sin(\theta), R\cos(\theta), m\theta)$$

then we calculate,

$$\overline{T} = \frac{1}{\sqrt{R^2+m^2}}\left[R\cos\theta\, U_1 - R\sin\theta\, U_2 + m\, U_3\right] \qquad \& \qquad \overline{N} = -\sin\theta\, U_1 - \cos\theta\, U_2$$

Thus, $\overline{B} = \overline{T} \times \overline{N}$ *yields*

$$\overline{B} = \frac{1}{\sqrt{R^2+m^2}}\left[m\cos\theta\, U_1 - m\sin\theta\, U_2 - R\, U_3\right]$$

Differentiate, note $\theta' = 1/\sqrt{R^2+m^2}$ *hence by chain rule:*

$$\overline{B}' = \frac{1}{R^2+m^2}\left[-m\sin\theta\, U_1 - m\cos\theta\, U_2\right] \quad \Rightarrow \quad \bar{\tau} = -\overline{B}' \bullet \overline{N} = \frac{-m}{m^2+R^2}.$$

Notice that the torsion of $\bar{\alpha}$ is the negative of that which we found in Example 2.4.4. The helix $\bar{\alpha}$ follows a left-hand-rule whereas the original helix α winds in a right-handed fashion. Swapping sine and cosine amounts to changing from a clockwise to a counter-clockwise winding as viewed from the positive z-axis. If you consider the $T \times N$ verses $\overline{T} \times \overline{N}$ geometrically it is easy to see B points up whereas $\overline{B}$ points down relative to the positive z-axis. But, both α and $\bar{\alpha}$ have a positive z-velocity hence the curve bends up off the osculating plane in the α case but, it bends down off the osculating plane in the $\bar{\alpha}$ case.

Following O'neill, we define two curves to be congruent if they have the same speed and shape. This is concisely given by:

Definition 3.3.6. *congruence of parametrized curves*

> We say two parametrized curves $\alpha : I \to \mathbb{R}^3$ and $\beta : I \to \mathbb{R}^3$ are **congruent** if there exists an isometry F for which $\beta = F \circ \alpha$.

Since isometries generally have rather complicated formulas this allows us to twist given standard examples into nearly unrecognizable forms:

Example 3.3.7. *Consider the parametrized curve:*

$$\begin{aligned}\beta(t) = \big(\, & c_1 R\cos t + c_2 R\sin t + c_3 mt - 1 + 3/\sqrt{6}, \\ & c_3 R\cos t + c_3 R\sin t - 2c_3 mt - 2, \\ & c_2 R\cos t + c_1 R\sin t + c_3 mt - 3 - 3/\sqrt{6}\ \big)\end{aligned}$$

where $c_1 = 1/\sqrt{6} + 1/2$, $c_2 = 1/\sqrt{6} - 1/2$ and $c_3 = 1/\sqrt{6}$. In fact, this is just the **helix** *with radius R and slope m. To see this simply $x = R\cos t$, $y = R\sin t$ and $z = mt$ into the isometry*

$$F(x, y, z) = \left[\begin{array}{c|c|c} \frac{1}{\sqrt{6}} + \frac{1}{2} & \frac{1}{\sqrt{6}} - \frac{1}{2} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} - \frac{1}{2} & \frac{1}{\sqrt{6}} + \frac{1}{2} & \frac{1}{\sqrt{6}} \end{array}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 + \frac{3}{\sqrt{6}} \\ 2 \\ 3 + \frac{3}{\sqrt{6}} \end{bmatrix}.$$

as to obtain $\beta(t)$ by calculating $F(\alpha(t))$.

You can see identifying standard curves simply by their formulas is generally a formidable task.

Definition 3.3.8. *parallel curves*

> We say two parametrized curves $\alpha : I \to \mathbb{R}^3$ and $\beta : I \to \mathbb{R}^3$ are **parallel** if there exists $p \in \mathbb{R}^3$ for which $\beta(t) = \alpha(t) + p$ for all $t \in I$.

Part (i.) of Theorem 2.3.8 told us $\gamma'(t) = 0$ for all $t \in I$ iff $\gamma(t) = p$ for all $t \in I$. If we study $\gamma(t) = \beta(t) - \alpha(t)$ then the proposition below follows naturally from Theorem 2.3.8.

Proposition 3.3.9.

> Parametrized curves $\alpha, \beta : I \to \mathbb{R}^3$ are parallel if and only if $\alpha'(t) = \beta'(t)$ for all $t \in I$. Moreover, if α and β are parallel and there exists t_o for which $\alpha(t_o) = \beta(t_o)$ then $\alpha = \beta$.

Proof: suppose $\alpha, \beta : I \to \mathbb{R}^3$ are parallel hence $\beta(t) = \alpha(t) + p$ for all $t \in I$. Differentiate to see $\beta'(t) = \alpha'(t)$ for all $t \in I$. Conversely, suppose $\beta'(t) = \alpha'(t)$ for all $t \in I$. Thus $\frac{d\alpha^i}{dt} = \frac{d\beta^i}{dt}$ for $i = 1, 2, 3$. Integrate with respect to t to obtain $\alpha^i(t) = \beta^i(t) + p^i$ for $i = 1, 2, 3$ where $p^i \in \mathbb{R}$ is a constant. Thus, $\alpha(t) = \beta(t) + p$ for each $t \in I$. Finally, suppose $\alpha, \beta : I \to \mathbb{R}^3$ are parallel with $\alpha(t_o) = \beta(t_o)$. But, $\beta(t) = \alpha(t) + p$ also implies $\beta(t_o) = \alpha(t_o) + p$ hence $\alpha(t_o) = \alpha(t_o) + p$ thus $p = 0$ and we conclude $\beta = \alpha$. $\square$

This definition of parallel covers more than just lines. The usual theorems for parallel lines in euclidean space clearly do not apply:

Example 3.3.10. *If $\alpha(t) = (R\cos t, R\sin t, 0)$ and $\beta(t) = (a + R\cos t, b + R\sin t, c)$ then $\beta(t) = \alpha(t) + (a, b, c)$. These are parallel circles. If $c = 0$ and a, b are sufficiently small, these parallel circles can be distinct and yet intersect.*

Finally we come to the central result of this Chapter. Equivalence of curvature and torsion provides equivalence of curves. Here we think of two arclength parametrized curves as equivalent if they are congruent. You can show that congruence is an equivalence relation. So, another way to understand the theorem which follows is that the curvature and torsion functions serve as labels for the equivalence class of unit-speed curves related by rigid motions.

Theorem 3.3.11. *curves classified by curvature and torsion in $\mathbb{R}^3$*

Arclength parametrized curves $\alpha, \bar{\alpha} : I \to \mathbb{R}^3$ are congruent iff $\bar{\kappa} = \kappa$ and $\bar{\tau} = \pm\tau$.

Proof: Theorem 3.3.4 shows congruent curves have the same curvature and torsion modulo a sign. It remains to prove the converse direction. Suppose $\alpha, \bar{\alpha} : I \to \mathbb{R}^3$ are arclength parametrized curves for which $\bar{\kappa} = \kappa$ and $\bar{\tau} = \pm\tau$. Suppose $s_o \in I$ and $\bar{\tau} = \tau$ define F to be the unique isometry which pushes the Frenet frame of α at $\alpha(s_o)$ to the Frenet frame of $\bar{\alpha}$ at $\bar{\alpha}(s_o)$:

$$\overline{T}_o = F_*(T_o), \quad \overline{N}_o = F_*(N_o), \quad \overline{B}_o = F_*(B_o), \quad F(\alpha(s_o)) = \bar{\alpha}(s_o) \quad \star\,(s_o)$$

where $T_o = T(s_o)$ etc. Notice, I only claim initially that F_* matches the Frenet frames at $s = s_o$. We know this is possible by Theorem 3.3.1. Apply Theorem 3.3.4 to show $F_*(T), F_*(N), F_*(B)$ solve the Frenet Serret equations with κ and τ coefficients. Therefore, we have Frenet Serret equations for the $F \circ \alpha$ Frenet frame and the $\bar{\alpha}$ Frenet frame: let me set $\bar{\kappa} = \kappa$ and $\bar{\tau} = \tau$ for our future calculational convenience:

$$\begin{array}{lcl} F_*(T)' = \kappa\, F_*(N) & & \overline{T}' = \kappa\,\overline{N} \\ F_*(N)' = -\kappa\, F_*(T) + \tau\, F_*(B) & \& & \overline{N}' = -\kappa\,\overline{T} + \tau\,\overline{B} \\ F_*(B)' = -\tau F_*(N) & & \overline{B}' = -\tau\overline{N} \end{array} \qquad (3.5)$$

We wish to show $F \circ \alpha = \bar{\alpha}$. Notice, as $(F \circ \alpha)' = F_*(T)$ and $\bar{\alpha}' = \overline{T}$ to show $F \circ \alpha$ and $\bar{\alpha}$ are parallel we must show $F_*(T)$ and $\overline{T}$ have the same vector parts. But, this is nicely captured by the dot-product $F_*(T) \bullet \overline{T} = 1$ since both $||F_*(T)|| = ||\overline{T}|| = 1$. Moreover, we know $F_*(T(s_o)) = \overline{T}(s_o)$ by construction of F hence we simply need to show that the function $g(s) = F_*(T(s)) \bullet \overline{T}(s)$ has constant value of 1. It turns out that is not directly successful, so, we instead follow O'neill[13] and involve the entire Frenet frames. Set

$$f(s) = F_*(T(s)) \bullet \overline{T}(s) + F_*(N(s)) \bullet \overline{N}(s) + F_*(B(s)) \bullet \overline{B}(s)$$

Omit the s-dependence and differentiate:

$$f' = F_*(T)' \bullet \overline{T} + F_*(T) \bullet \overline{T}' + F_*(N)' \bullet \overline{N} + F_*(N) \bullet \overline{N}' + F_*(B)' \bullet \overline{B} + F_*(B) \bullet \overline{B}'$$

[13] beware, my $\bar{\alpha}$ is his β on page 122-123

Subtitute Equation 3.5 to obtain:

$$\begin{aligned} f' =& \kappa\, F_*(N) \bullet \overline{T} + F_*(T) \bullet \kappa\overline{N} + \big(-\kappa\, F_*(T) + \tau\, F_*(B)\big) \bullet \overline{N} \\ & + F_*(N) \bullet \big(-\kappa\,\overline{T} + \tau\,\overline{B}\big) + -\tau F_*(N) \bullet \overline{B} + F_*(B) \bullet \big(-\tau\overline{N}\big) = 0 \end{aligned}$$

But, $f(s_o) = F_*(T(s_o)) \bullet \overline{T}(s_o) + F_*(N(s_o)) \bullet \overline{N}(s_o) + F_*(B(s_o)) \bullet \overline{B}(s_o) = 3$ thus all three dot-products must be identically 1 and we find that $F \circ \alpha$ and $\bar{\alpha}$ are parallel with $(F \circ \alpha)(s_o) = \bar{\alpha}(s_o)$ thus, by Proposition 3.3.9, we conclude $F \circ \alpha = \bar{\alpha}$. Hence α and $\bar{\alpha}$ are congruent in the case $\bar{\tau} = \tau$. I leave the $\bar{\tau} = -\tau$ case to the reader. $\square$

This theorem easily generalizes to the case of non-unit speed. The derivation just includes a few extra speed factors. The Proposition below exhibits the power of the theorem towards questions of classification:

Proposition 3.3.12.

If α be an arclength parametrized curve then α has constant κ, τ iff α is a helix.

Proof: In Example 2.4.4 we introduced the standard helix defined by $R, m > 0$ and

$$\alpha(s) = (R\cos(ks), R\sin(ks), mks)$$

for $s \in \mathbb{R}$ and $k = 1/\sqrt{R^2 + m^2}$. We calculated:

$$\kappa = \frac{R}{R^2 + m^2} \qquad \& \qquad \tau = \frac{m}{R^2 + m^2}$$

If F is an isometry then $F \circ \alpha$ is also a helix and by Theorem 3.3.4 the non-standard helix also has $\kappa = \frac{R}{R^2+m^2}$ and $\tau = \frac{\pm m}{R^2+m^2}$. Conversely, suppose α is an arclength parametrized curve with constant κ and τ. Observe, if we set values for R and m as

$$R = \frac{\kappa}{\kappa^2 + \tau^2}, \qquad \& \qquad m = \frac{\tau}{\kappa^2 + \tau^2}$$

then

$$\frac{R}{R^2 + m^2} = \frac{\frac{\kappa}{\kappa^2+\tau^2}}{\frac{\kappa^2}{(\kappa^2+\tau^2)^2} + \frac{\tau^2}{(\kappa^2+\tau^2)^2}} = \kappa \qquad \& \qquad \frac{m}{R^2 + m^2} = \frac{\frac{\tau}{\kappa^2+\tau^2}}{\frac{\kappa^2}{(\kappa^2+\tau^2)^2} + \frac{\tau^2}{(\kappa^2+\tau^2)^2}} = \tau.$$

Hence, by Theorem 3.3.11 the curve α with constant κ and τ is congruent to the helix with $R = \frac{\kappa}{\kappa^2+\tau^2}$ and $m = \frac{\tau}{\kappa^2+\tau^2}$ since the helix and α have the same curvature and torsion. $\square$

Also, the reader may enjoy Example 5.4 on page 124 of O'neill which explicitly shows how a pair of helices with opposite-signed torsion are connected by an isometry with an orthogonal matrix with determinant -1.

The result below is needed for Theorem 9.2 on page 316-317 of O'neill where this result is critical to prove something about the isometry of surfaces. The proof is placed here since the technique is very much reminiscent of proof given for Theorem 3.3.11. Notice, the frames in the theorem below are not assumed to be Frenet frames. The assignment of the frame along the curve could be made by some other set of rules.

Theorem 3.3.13. *curve congruence for arbitrary frames*

Suppose $\alpha, \beta : I \to \mathbb{R}^3$ are parametrized curves. Also, let E_1, E_2, E_3 be a frame field along α and F_1, F_2, F_3 a frame field along β. If the two conditions below hold true then α and β are congruent. For $1 \leq i, j \leq 3$,

$$(1.)\ \alpha' \bullet E_i = \beta' \bullet F_i \qquad \& \qquad (2.)\ E_i' \bullet E_j = F_i' \bullet F_j.$$

In particular, in the case these conditions are met, the isometry G for which $\beta = G \circ \alpha$ is defined by any $t_o \in I$ where we construct $G_*(E_i(\alpha(t_o))) = F_i(\beta(t_o))$ for $i = 1, 2, 3$.

Proof: see page 126 of O'neill. $\square$

Chapter 4

Calculus of Surfaces

In this Chapter we follow Oneill's Chapter 4 and discuss calculus on a surface