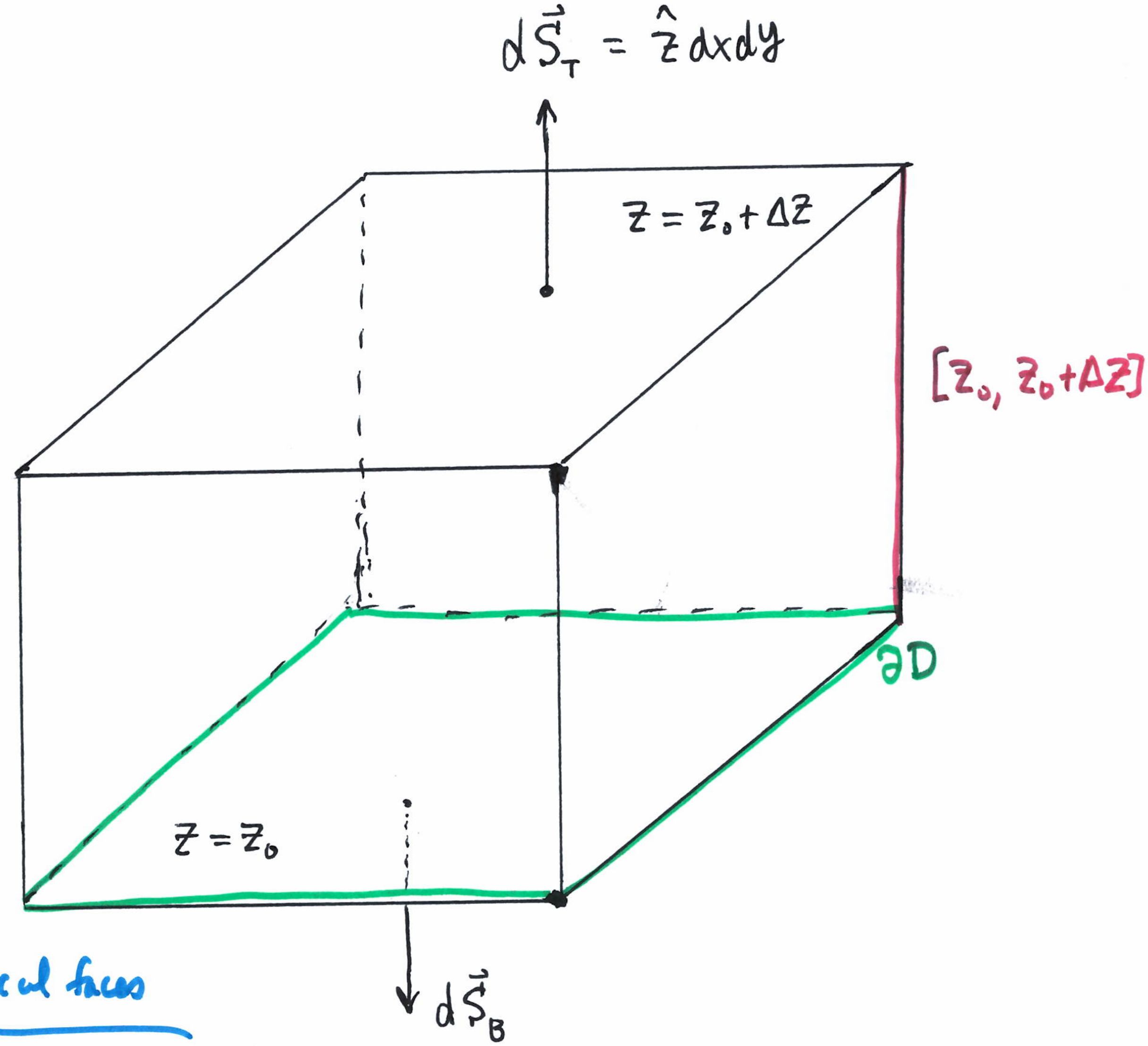
DIVERGENCE THEOREM FOR RECTANGULAR SLAB



$$\underbrace{\partial D \times [z_0, z_0+\Delta z]}_{\text{vertical faces}}$$

$$\Phi_{\text{vertical faces}} \cong \Delta z \int_{\partial D} \langle P, Q, R\rangle \cdot \hat{n}\, ds \qquad (\text{but } \hat{n}\cdot\hat{z} = 0)$$

$$= \Delta z \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

$$= \iiint_{D\times[z_0,\, z_0+\Delta z]} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dV$$

$$\Phi_T + \Phi_B = \iint_T \langle P, Q, R\rangle \cdot (\hat{z}\, dx\,dy) + \iint_B \langle P, Q, R\rangle \cdot (-\hat{z}\, dx\,dy)$$

$$= \iint_D \Big(R(x, y, z_o + \Delta z) - R(x, y, z_o)\Big)\, dx\,dy$$

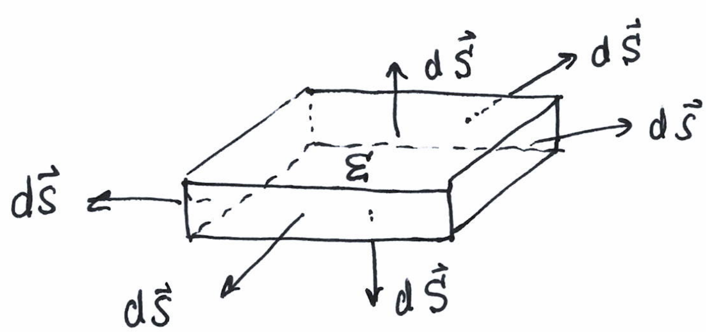
$$= \iint_D \left(\frac{\partial R}{\partial z}(x, y, z^*)\, \Delta z\right) dx\,dy \qquad \underbrace{z^* \in [z_o, z_o + \Delta z)}_{\text{By Mean Value Th}^{\underline{m}}}$$

$$\cong \iiint_{D \times [z_o, z_o + \Delta z]} \left(\frac{\partial R}{\partial z}\right) dz\, dx\, dy$$

Then, as $\Delta z \longrightarrow 0$ we obtain for $\mathcal{E} = D \times [z_o, z_o + \Delta z]$

$$\Phi_{TOTAL} = \iint_{\partial \mathcal{E}} \langle P, Q, R\rangle \cdot d\vec{S} = \iiint_{\mathcal{E}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV$$

$$\Rightarrow \boxed{\iint_{\partial \mathcal{E}} \vec{F} \cdot d\vec{S} = \iiint_{\mathcal{E}} (\nabla \cdot \vec{F})\, dV}$$

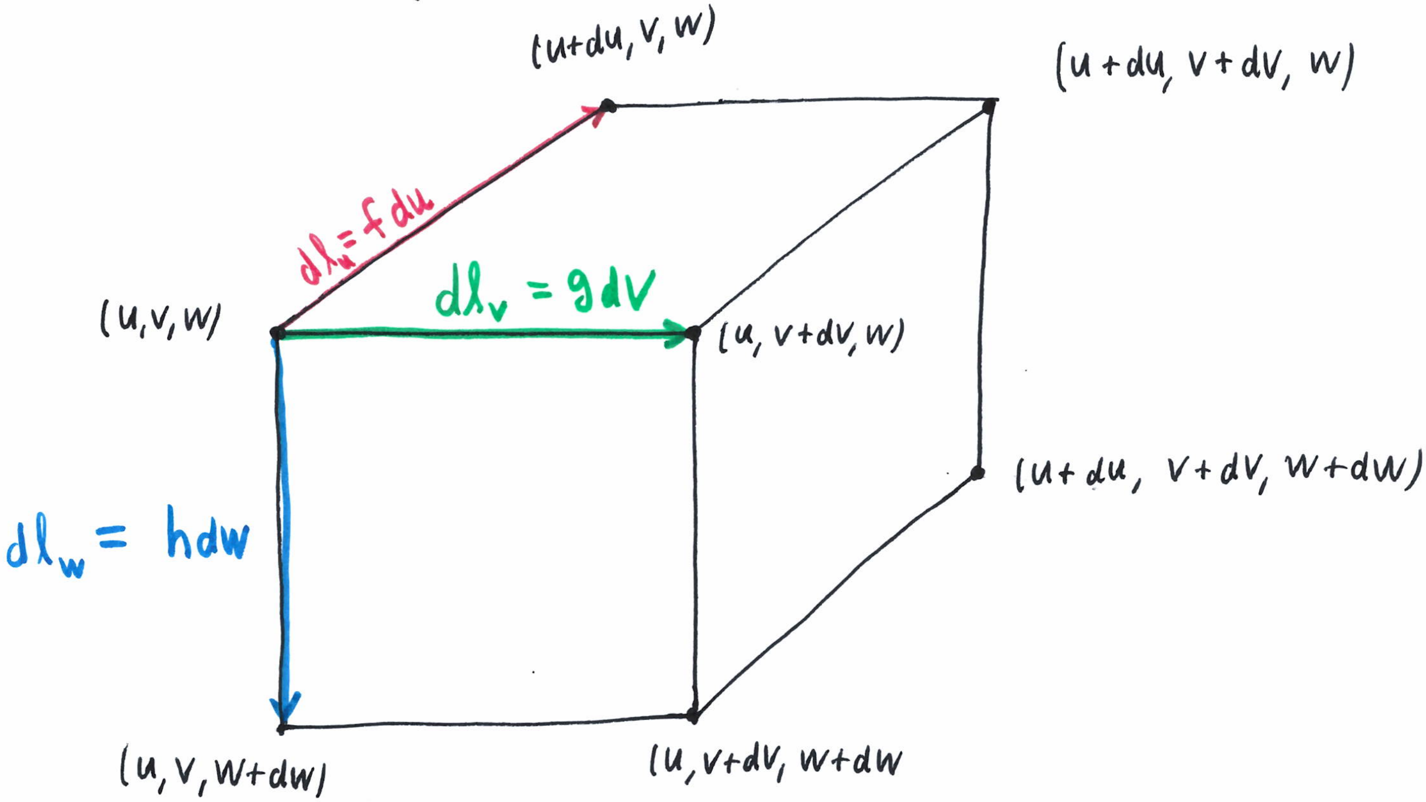


($\partial \mathcal{E}$ outward)

Remark: what follows is an infinitesimal justification for the divergence and curl formulas in arbitrary orthogonal coord. In my old handwritten notes, you can find gory brute force derivations of the same for cylindrical and spherical coord.

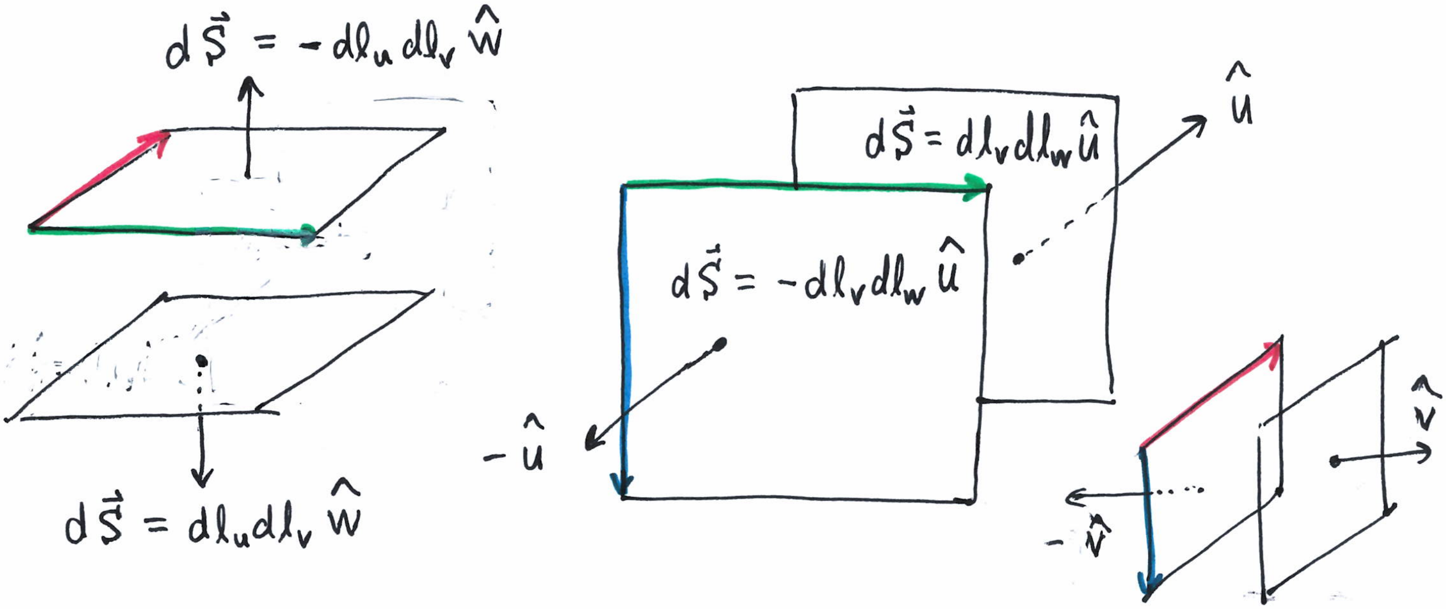
Imagine an infinitesimal cube in u, v, w – coordinate grid

We wish to analyze $\vec{A}(u,v,w) = A_u \hat{u} + A_v \hat{v} + A_w \hat{w}$ to find $\oint \vec{A} \cdot d\vec{S}$



Assuming u, v, w are orthogonal coordinates $d\vec{l}_u \perp d\vec{l}_v \perp d\vec{l}_w$

Thus $dV = dl_u\, dl_v\, dl_w = f g h\, du\, dv\, dw$. We assume $du, dv, dw > 0$ hence $\pm\nabla u, \pm\nabla v, \pm\nabla w$ point normal to faces of cube. In particular,



Consider u and $u+du$ faces

front | $\vec{A}\cdot d\vec{S} = (A_u\hat{u} + A_v\hat{v} + A_w\hat{w})\cdot(-gh\,dv\,dw\,\hat{u})$

back | $\vec{A}\cdot d\vec{S} = (A_u\hat{u} + A_v\hat{v} + A_w\hat{w})\cdot(gh\,dv\,dw\,\hat{u})$

Total from fixed-u faces, let F

$$\Phi_u = \left[(ghA_u)(u+du,v,w) - (ghA_u)(u,v,w)\right]dv\,dw$$

$$= \frac{\partial}{\partial u}\left[ghA_u\right]du\,dv\,dw$$

$$= \underline{\frac{1}{fgh}\frac{\partial}{\partial u}\left[ghA_u\right]dV} \quad \mathrm{I}$$

Likewise, from fixed-v faces,

$$\Phi_v = \left[(fhA_v)(u,v+dv,w) - (fhA_v)(u,v,w)\right]du\,dw$$

$$= \frac{\partial}{\partial v}\left[fhA_v\right]du\,dv\,dw$$

$$= \underline{\frac{1}{fgh}\frac{\partial}{\partial v}\left[fhA_v\right]dV} \quad \mathrm{II}$$

And finally, for fixed-w faces,

$$\Phi_w = \underline{\frac{1}{fgh}\frac{\partial}{\partial w}\left[fgA_w\right]dV} \quad \mathrm{III}$$

We find

$$\oint \vec{A}\cdot d\vec{S} = \underbrace{\frac{1}{fgh}\left[\frac{\partial}{\partial u}\left[ghA_u\right] + \frac{\partial}{\partial v}\left[fhA_v\right] + \frac{\partial}{\partial w}\left[fgA_w\right]\right]}_{\text{formula for } \nabla\cdot\vec{A} \text{ in } u,v,w\text{-coordinates!}}dV$$

Cylindrical Coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$dx^2 + dy^2 + dz^2 = \underbrace{dr^2 + r^2 d\theta^2 + dz^2}$$

$$l_r = 1, \quad l_\theta = r, \quad l_z = 1$$
$$f = 1 \quad g = r \quad h = 1$$

If $\vec{F} = F_r \hat{r} + F_\theta \hat{\theta} + F_z \hat{z}$ then,

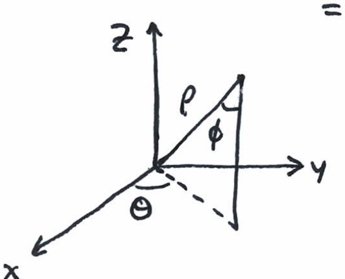
$$\nabla \cdot \vec{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left[r A_r\right] + \frac{\partial}{\partial \theta}\left[A_\theta\right] + \frac{\partial}{\partial z}\left[r A_z\right]\right]$$

$$\boxed{\nabla \cdot \vec{F} = \frac{1}{r}\frac{\partial}{\partial r}\left[r A_r\right] + \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}}$$

Spherical Coordinates

$$x = \rho\cos\theta\sin\phi \qquad y = \rho\sin\theta\sin\phi \qquad z = \rho\cos\phi$$

$$dx^2 + dy^2 + dz^2 = d\rho^2 + r^2 d\theta^2 + \rho^2 d\phi^2$$
$$= d\rho^2 + \rho^2\sin^2\phi \, d\theta^2 + \rho^2 d\phi^2$$



$$\hat{\rho} \times \hat{\phi} = \hat{\theta}$$

$$l_\rho = 1, \quad l_\phi = \rho, \quad l_\theta = \rho\sin\phi$$
$$f = 1, \quad g = \rho, \quad h = \rho\sin\phi$$

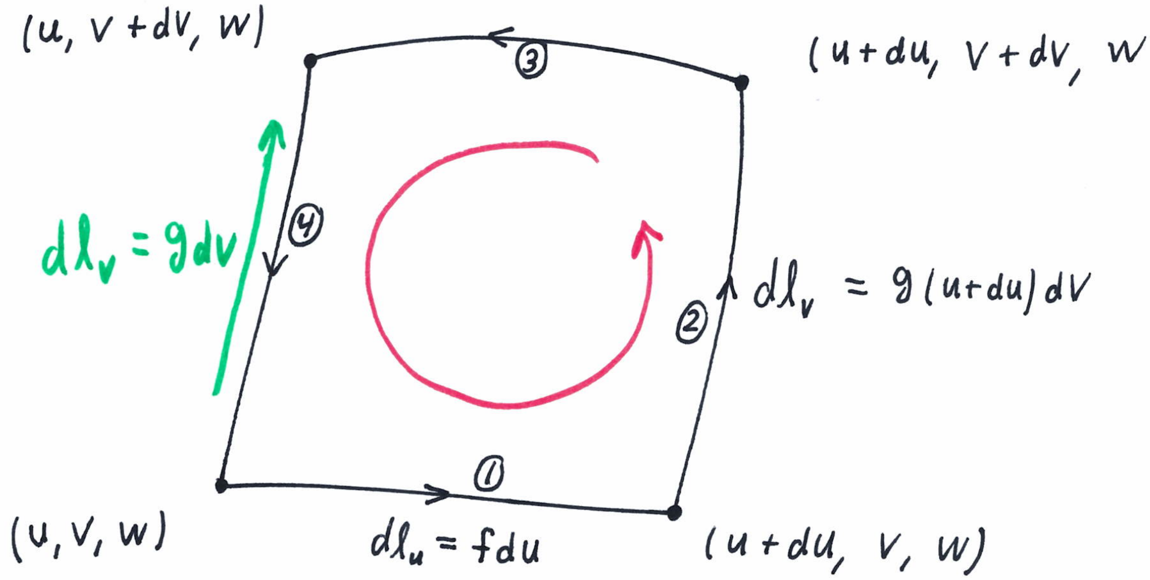
For $\vec{F} = F_\rho \hat{\rho} + F_\phi \hat{\phi} + F_\theta \hat{\theta}$,

$$\nabla \cdot \vec{F} = \frac{1}{\rho^2\sin\phi}\left[\frac{\partial}{\partial \rho}\left[\rho^2\sin\phi \, F_\rho\right] + \frac{\partial}{\partial \phi}\left[\rho\sin\phi \, F_\phi\right] + \frac{\partial}{\partial \theta}\left[\rho F_\theta\right]\right]$$

$$\boxed{\nabla \cdot \vec{F} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left[\rho^2 F_\rho\right] + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial \phi}\left[\sin\phi \, F_\phi\right] + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial \theta}\left[F_\theta\right]}$$

DERIVATION OF $\nabla \times \vec{A}$ in u, v, w orthogonal coordinates

Imagine a little patch with fixed-w, so $\hat{w}$ is normal,



Observe $d\vec{S} = d\vec{l}_u \times d\vec{l}_v = (fg\, du\, dv)\,\hat{w}$.

Calculate the circulation of $\vec{A} = A_u\hat{u} + A_v\hat{v} + A_w\hat{w}$ around the edge,

① $\vec{A}\cdot d\vec{r} = \vec{A}\cdot(f\,du\,\hat{u}) = A_u f\,du$ (at fixed v, w)

③ $\vec{A}\cdot d\vec{r} = -\vec{A}\cdot(f\,du\,\hat{u}) = -A_u f\,du$ (at fixed $v+dv, w$)

From ① and ③,

$$\vec{A}\cdot d\vec{r}\Big|_{①} + \vec{A}\cdot d\vec{r}\Big|_{③} = \Big[(fA_u)(u,v,w) - (fA_u)(u,v+dv,w)\Big]du$$
$$= -\frac{\partial}{\partial v}\left[fA_u\right]dv\,du$$

Likewise, from ② and ④,

$$\vec{A}\cdot d\vec{r}\Big|_{②} + \vec{A}\cdot d\vec{r}\Big|_{④} = \Big[(gA_v)(u+du,v,w) - (gA_v)(u,v,w)\Big]dv$$
$$= \frac{\partial}{\partial u}\left[gA_v\right]du\,dv$$

$$\oint \vec{A}\cdot d\vec{r} = \left[\frac{\partial}{\partial u}\left[gA_v\right] - \frac{\partial}{\partial v}\left[fA_u\right]\right]du\,dv$$
$$= \frac{1}{fg}\left[\frac{\partial}{\partial u}[gA_v] - \frac{\partial}{\partial v}[fA_u]\right]\hat{w}\cdot d\vec{S} = (\nabla\times\vec{A})_w\,\hat{w}\cdot d\vec{S}$$

We found for $\vec{A} = A_u \hat{u} + A_v \hat{v} + A_w \hat{w}$

$$\hat{w} \cdot (\nabla \times \vec{A}) = \frac{1}{fg}\left[\frac{\partial}{\partial u}[g A_v] - \frac{\partial}{\partial v}[f A_u]\right] \quad \mathrm{I}$$

By symmetry of u, v, w coordinates right-handed interchange,

$u, v, w \longrightarrow v, w, u$ or $u, v, w \longrightarrow w, u, v$

$f, g, h \longrightarrow g, h, f$ $\quad f, g, h \longrightarrow h, f, g$

we deduce

$$\hat{u} \cdot (\nabla \times \vec{A}) = \frac{1}{gh}\left[\frac{\partial}{\partial v}[h A_w] - \frac{\partial}{\partial w}[g A_v]\right] \quad \mathrm{II}$$

Likewise,

$$\hat{v} \cdot (\nabla \times \vec{A}) = \frac{1}{hf}\left[\frac{\partial}{\partial w}[f A_u] - \frac{\partial}{\partial u}[h A_w]\right] \quad \mathrm{III}$$

From I, II, III we find:

$$\boxed{\begin{aligned}\nabla \times \vec{A} = {} & \frac{\hat{u}}{gh}\left[\frac{\partial}{\partial v}[hA_w] - \frac{\partial}{\partial w}[gA_v]\right] \\ & + \frac{\hat{v}}{hf}\left[\frac{\partial}{\partial w}[fA_u] - \frac{\partial}{\partial u}[hA_w]\right] \\ & + \frac{\hat{w}}{fg}\left[\frac{\partial}{\partial u}[gA_v] - \frac{\partial}{\partial v}[fA_u]\right]\end{aligned}}$$

$$\boxed{\nabla \times \vec{A} = \det \begin{bmatrix} \frac{1}{gh}\hat{u} & \frac{1}{hf}\hat{v} & \frac{1}{fg}\hat{w} \\ \partial_u & \partial_v & \partial_w \\ fA_u & gA_v & hA_w \end{bmatrix}}$$

FORMULAS FOR CURL IN uvw - coordinates

Given ~~$\vec{F}$~~ $\vec{A} = A_u \hat{u} + A_v \hat{v} + A_w \hat{w}$ we can argue by examining infinitesimal fixed u, v, w faces that it follows

$$\boxed{\begin{aligned}\nabla \times \vec{A} = &\frac{1}{gh}\left[\frac{\partial}{\partial v}(h A_w) - \frac{\partial}{\partial w}(g A_v)\right]\hat{u} \\ &+ \frac{1}{fh}\left[\frac{\partial}{\partial w}(f A_u) - \frac{\partial}{\partial u}(h A_w)\right]\hat{v} \\ &+ \frac{1}{fg}\left[\frac{\partial}{\partial u}(g A_v) - \frac{\partial}{\partial v}(f A_u)\right]\hat{w}\end{aligned}}$$

Cylindricals: $(\overset{u}{r}, \overset{v}{\theta}, \overset{w}{z})$ with $f=1$, $g=r$, $h=1$

$$\begin{aligned}\nabla \times \vec{A} = &\frac{1}{r}\left[\frac{\partial}{\partial \theta}\left[A_z\right] - \frac{\partial}{\partial z}\left[r A_\theta\right]\right]\hat{r} \\ &+ \left[\frac{\partial}{\partial z}\left[A_r\right] - \frac{\partial}{\partial r}\left[A_z\right]\right]\hat{\theta} \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}\left[r A_\theta\right] - \frac{\partial}{\partial \theta}\left[A_r\right]\right]\hat{z}\end{aligned}$$

$$\boxed{\nabla \times \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(A_\theta r) - \frac{\partial A_r}{\partial \theta}\right)\hat{z}}$$

Sphericals: (ρ, ϕ, θ) with $f=1$, $g=\rho$, $h=\rho\sin\phi$

Curl in Spherical Coordinates

Let $\vec{A} = A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_\theta \hat{\theta}$ then $f=1,\ g=\rho,\ h=\rho\sin\phi$

$$\nabla \times \vec{A} = \det \begin{bmatrix} \frac{1}{\rho^2 \sin\phi}\hat{\rho} & \frac{1}{\rho\sin\phi}\hat{\phi} & \frac{1}{\rho}\hat{\theta} \\ \partial_\rho & \partial_\phi & \partial_\theta \\ A_\rho & \rho A_\phi & \rho\sin\phi A_\theta \end{bmatrix}$$

$$= \frac{\hat{\rho}}{\rho^2\sin\phi}\left[\frac{\partial}{\partial\phi}[\rho\sin\phi A_\theta] - \frac{\partial}{\partial\theta}[\rho A_\phi]\right]$$

$$- \frac{\hat{\phi}}{\rho\sin\phi}\left[\frac{\partial}{\partial\rho}[\rho\sin\phi A_\theta] - \frac{\partial}{\partial\theta}[A_\rho]\right]$$

$$+ \frac{\hat{\theta}}{\rho}\left[\frac{\partial}{\partial\rho}[\rho A_\phi] - \frac{\partial}{\partial\phi}[A_\rho]\right]$$

$$= \frac{1}{\rho\sin\phi}\left(\frac{\partial}{\partial\phi}[\sin\phi A_\theta] - \frac{\partial A_\phi}{\partial\theta}\right)\hat{\rho}$$

$$- \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}[\rho A_\theta] - \frac{1}{\rho\sin\phi}\frac{\partial A_\rho}{\partial\theta}\right)\hat{\phi}$$

$$+ \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}[\rho A_\phi] - \frac{\partial A_\rho}{\partial\phi}\right)\hat{\theta}$$

$$\boxed{\nabla \times \vec{A} = \frac{1}{\rho\sin\phi}\left[\frac{\partial}{\partial\phi}[\sin\phi A_\theta] - \frac{\partial A_\phi}{\partial\theta}\right]\hat{\rho} + \frac{1}{\rho}\left[\frac{1}{\sin\phi}\frac{\partial A_\rho}{\partial\theta} - \frac{\partial}{\partial\rho}[\rho A_\theta]\right]\hat{\phi} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}[\rho A_\phi] - \frac{\partial A_\rho}{\partial\phi}\right]\hat{\theta}}$$

Remark: pg. 8 of my mostly handwritten p. 375-384 notes agrees.