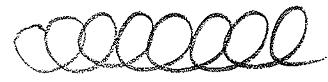
INDUCTANCE

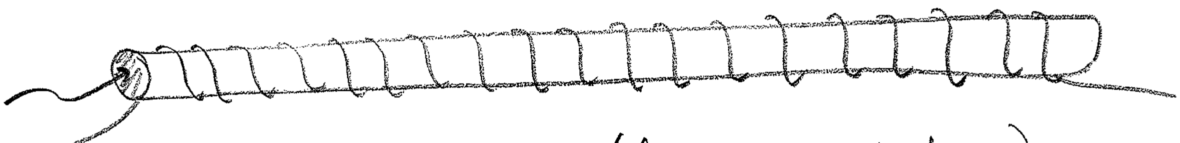
The magnetic flux through a coil  is proportional to the current I traveling through the coil. We define self-inductance with that in mind,

$$\underbrace{\Phi_B}_{\text{flux in coil}} = \underbrace{L}_{\text{proportionality constant}} \, \underbrace{I}_{\text{current}}$$

L is called the inductance, it's units are H for Henry

E79 For a solenoid (long) with area A and $n = \frac{\text{turns}}{\text{length}}$

A



$$B = \mu_0 n I \quad \text{(from ampere's Law)}$$

$$\Phi_B = NBA = N(\mu_0 n I)A \quad \text{where } n = \frac{N}{l}$$

total turns (N), length (l)

$$L = \frac{\Phi_B}{I} = \frac{\mu_0 n N I A}{I}$$

$$\Rightarrow L = \mu_0 n N A \quad (\text{note } N = nl)$$

$$\Rightarrow \boxed{L = \mu_0 n^2 A l}$$

The reason for the "N" in Φ_B is that the flux goes through each and every one of the N-loops which make up the solenoid.

Remark: $\mu_0 = 4\pi \times 10^{-7}$ H/m

Inductors and Circuits

$$\Phi_B = L I$$

$$\frac{d\Phi_B}{dt} = \frac{d}{dt}(LI) = L\frac{dI}{dt}$$

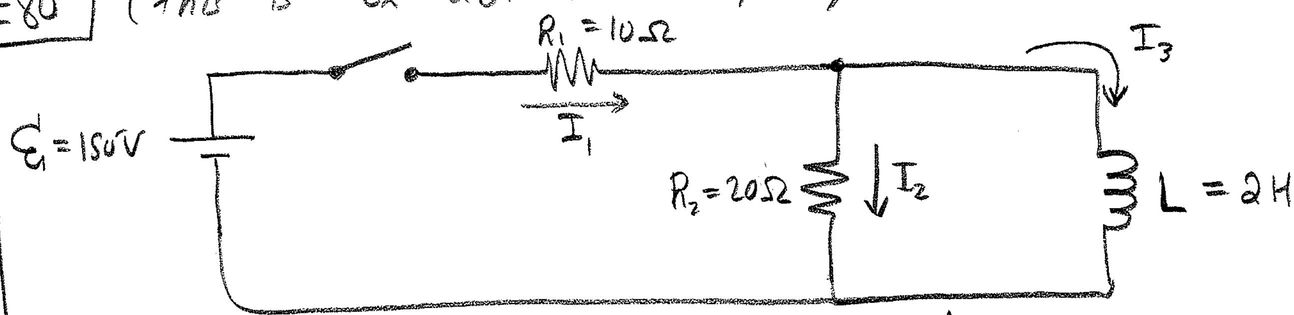
for most inductors, the geometry is fixed so L doesn't change with time.

But, $\frac{d\Phi_B}{dt} = -\mathcal{E}$ $\therefore$ $\underline{\mathcal{E} = -L\frac{dI}{dt}}$ (resists instant. change in current)

If the inductor has resistance r then

$$\underline{\Delta V = \mathcal{E} - Ir = -L\frac{dI}{dt} - Ir}.$$

E80 (this is Ex 28-14 in Tipler)



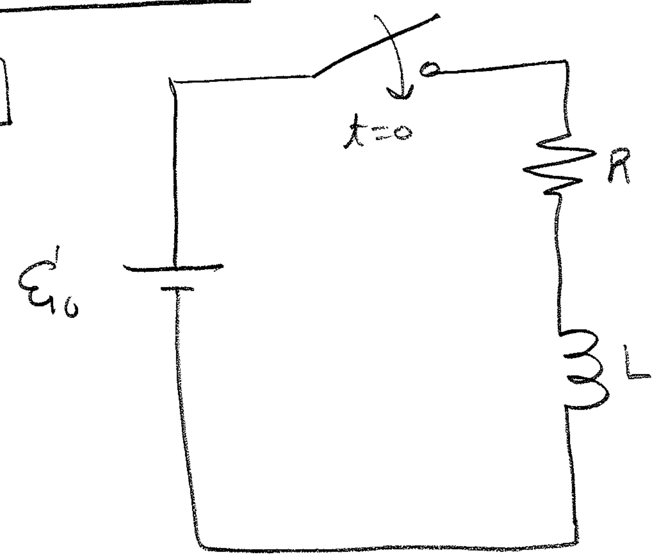
close switch at $t=0$ and solve the circuit

$$\left.\begin{aligned} \mathcal{E} - I_1R_1 - I_2R_2 &= 0 \\ I_2R_2 &= -L\frac{dI_3}{dt} \\ I_1 &= I_2 + I_3 \end{aligned}\right\}$$ challenge, solve.

(we'll analyze $t=0$ and $t \to \infty$ in lecture)

RL CIRCUIT

E81



close switch then find $I(t)$ (CHAD)

$$\mathcal{E}_0 - IR - L\frac{dI}{dt} = 0$$

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{\mathcal{E}_0}{L}$$

$$\Rightarrow \quad I(t) = c_1 e^{-\frac{Rt}{L}} + \frac{\mathcal{E}_0}{R}$$

(sep. of variables or const. coeff with annihilator method or $\int$-fact. method you do the math.)

$$I(0) = c_1 + \frac{\mathcal{E}_0}{R} = 0$$

initial current zero.

$$\therefore \quad c_1 = -\mathcal{E}_0/R$$

$$\Rightarrow \quad \boxed{I(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-t/\tau}\right)}$$ where $\underbrace{\tau = \frac{L}{R}}_{\text{time constant for RL circuit.}}$

The eq$^{\underline{n}}$ above is for $t \geq 0$ naturally $I(t) = 0$ for $t < 0$.

Remark: Mutual Inductance is defined for some pair of circuits. It describes how the current in circuit 1 creates a magnetic flux in circuit 2. Generically,

$$\underbrace{\Phi_B}_{\substack{\text{in circuit} \\ \#2 \\ \text{due to } I_1}} = \underbrace{M_{12}}_{\text{mutual inductance}} I_1$$

Likewise, one could define

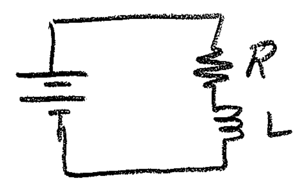
$$\underbrace{\Phi_B}_{\substack{\text{in circuit}\#1 \\ \text{due to } I_2}} = M_{21} I_2$$

I turns out that $M_{12} = M_{21}$ in general.

(I didn't assign any hwk on this)

Magnetic Energy

$$\mathcal{E}_0 - IR - L\frac{dI}{dt} = 0 \quad \text{for}$$



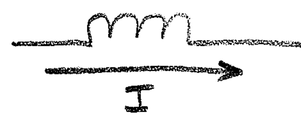
$$\Rightarrow \underbrace{\mathcal{E}_0 I}_{\substack{\text{rate energy} \\ \text{supplied} \\ \text{to circuit}}} = \underbrace{I^2R}_{\substack{\text{energy} \\ \text{lost in } R \\ \text{per time}}} + \underbrace{LI\frac{dI}{dt}}_{\text{energy per time stored in } L}$$

$$\frac{dU}{dt} = LI\frac{dI}{dt} \Rightarrow dU = LI\,dI$$

$$\Rightarrow \boxed{U = \frac{1}{2}LI^2}$$

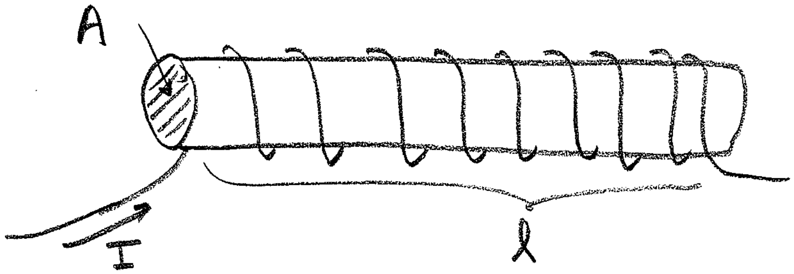
energy stored in inductor proportional to square of I flowing through I.

Energy Density in Magnetic Field

As I flows through  it sets-up a magnetic field with energy $U = \frac{1}{2} L I^2$.

For a solenoid we can calculate more:

$\boxed{\text{E82}}$ deriving energy density from Solenoid Example,



we derived this previous lecture.

$$B = \mu_0 n I \quad \text{and} \quad L = \mu_0 n^2 A l$$

$$\text{Volume} = A l$$

solving for B

$$\frac{U}{\text{Vol.}} = \frac{\frac{1}{2} L I^2}{A l} = \frac{\frac{1}{2}\mu_0 n^2 \cancel{A l}\, I^2}{\cancel{A l}} = \frac{1}{2}\mu_0 n^2 \left(\frac{B}{\mu_0 n}\right)^2$$

$$\therefore \boxed{\frac{\text{Energy in B-field}}{\text{Volume}} = \frac{1}{2\mu_0} B^2}$$

$\boxed{\text{E83}}$ In retro spect, a capacitor has

$$\boxed{\frac{\text{Energy in E-field}}{\text{Volume}} = \frac{1}{2}\epsilon_0 E^2}$$

To see this recall $U = \frac{1}{2} C V^2$ and $V = Ed$ and $C = \frac{A\epsilon_0}{d}$ and $\text{Vol} = Ad$ so

$$\frac{U}{\text{Vol}} = \frac{\frac{1}{2} C V^2}{Ad} = \frac{\frac{1}{2}\cancel{A}\epsilon_0}{\cancel{A}d^2}(Ed)^2 = \underline{\frac{1}{2}\epsilon_0 E^2}.$$

$\boxed{E82}$ & $\boxed{E83}$ actually reveal general formulas for energy density due to the Electric & Magnetic Fields.

You can derive the same $\frac{1}{2\mu_0}B^2$ or $\frac{\epsilon_0}{2}E^2$ energy dependence for constant B or E fields filling other geometries. Generally you have $\frac{dU}{dV} = \frac{1}{2\mu_0}B^2$ or $\frac{dU}{dV} = \frac{\epsilon_0}{2}E^2$

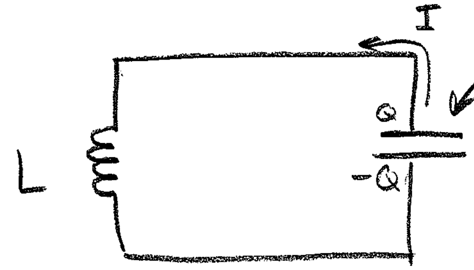
So to find total energy if B or E varies you do some volume integral

$$U_{\text{TOTAL IN } V} = \iiint_V \left(\frac{1}{2\mu_0}B^2 + \frac{\epsilon_0}{2}E^2\right) dV$$

(however since Calculus III is not prereq. we'll abstain from explicit computation of such integrals, fascinating as they may be...)

LC CIRCUIT

$\boxed{E84}$



initially charged to V_0
find $I(t)$ for $t > 0$.
(assume $I(0) = 0$.)

$$I = -\frac{dQ}{dt}$$

$$-L\frac{dI}{dt} + \frac{Q}{C} = 0$$

$$\Rightarrow -L\frac{d}{dt}\left(\frac{-dQ}{dt}\right) + \frac{Q}{C} = 0$$

$$\underbrace{\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0}$$

well-known DEq^n! $\longrightarrow$ $Q(t) = Q_0 \cos(\omega t + \phi_0)$, $\omega = \sqrt{\frac{1}{LC}}$

$$I(t) = -\frac{dQ}{dt} = \omega Q_0 \sin(\omega t + \phi_0)$$

Note, $V_0 = \frac{Q_0}{C} \Rightarrow \underline{Q_0 = V_0 C}$

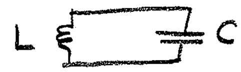
Also $I(0) = 0 = \omega Q_0 \sin(\phi_0) \Rightarrow$ <u>choose $\phi_0 = 0$</u>.

Hence $I(t) = \frac{V_0 C}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right)$

$$\boxed{I(t) = \omega V_0 C \sin(\omega t) \quad , \quad \omega = \frac{1}{\sqrt{LC}}}$$

Also, for $\boxed{E85}$ note $\boxed{Q(t) = C V_0 \cos(\omega t)}$

$\boxed{\text{E85}}$ Find energy in L or C at time t for E84 circuit



$$U_C = \frac{1}{2C} Q^2 = \frac{1}{2C}\left(C V_0 \cos(\omega t)\right)^2$$

$$\Rightarrow \boxed{U_C(t) = \frac{1}{2} C V_0^2 \cos^2(\omega t)}, \quad \omega = \frac{1}{\sqrt{LC}}$$

$$U_L = \frac{1}{2} L I^2 = \frac{1}{2} L \left(\omega V_0 C \sin(\omega t)\right)^2$$

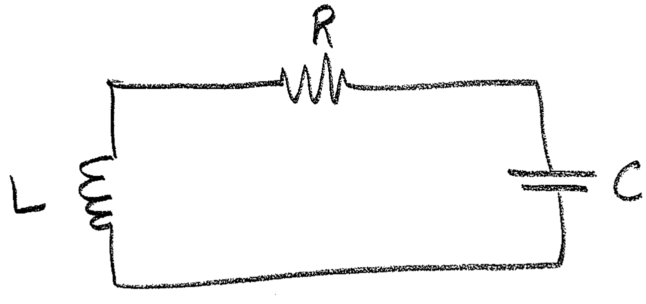
$$= \frac{1}{2} L C^2 \omega^2 V_0^2 \sin^2(\omega t)$$

$$= \frac{1}{2} \frac{LC^2}{LC} V_0^2 \sin^2(\omega t)$$

$$\Rightarrow \boxed{U_L(t) = \frac{1}{2} C V_0^2 \sin^2(\omega t)}.$$

Note $U_C + U_L = \frac{1}{2} C V_0^2$ which is precisely the initial energy stored in C. As time goes on the L & C simply pass the energy back and forth w/o end.

$\boxed{\text{E 86}}$



(like spring with friction, same math.)

$$L \frac{dI}{dt} + IR + \frac{Q}{C} = 0$$

$$\Rightarrow L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = 0$$

R=0, S.H.O.

critical damped

under damped