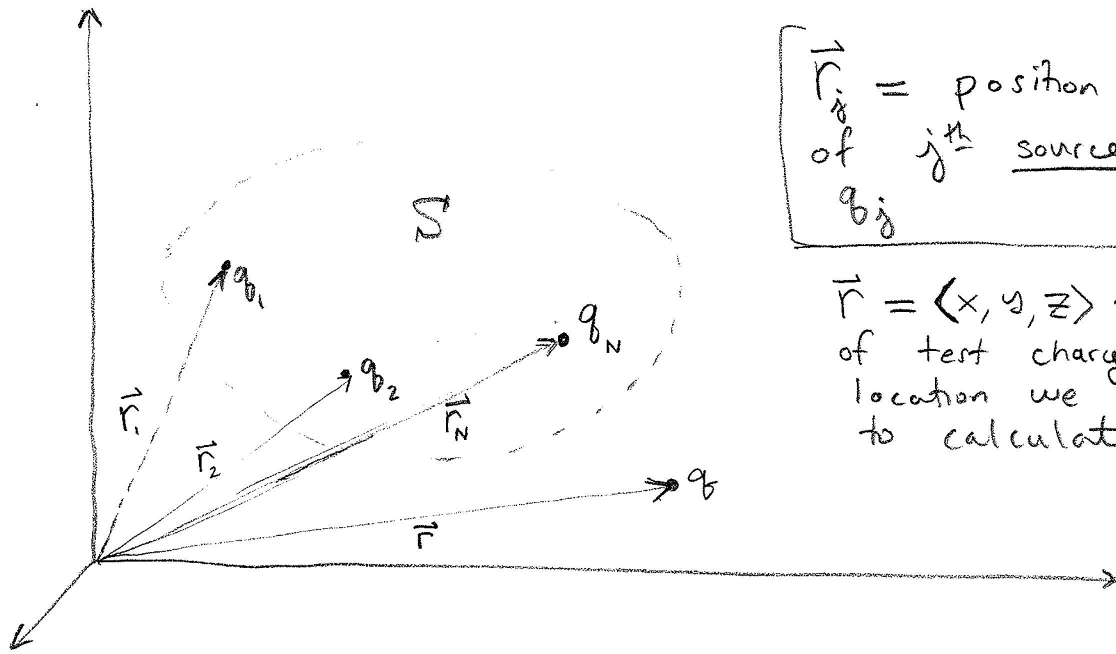
ELECTRIC FIELDS

Imagine some collection S of charges which is stuck in place at various positions. Then imagine introducing a tiny positive "test charge" q. As we place q at various positions in space we can calculate the net-force from S on q by taking the vector sum of the Coulomb forces produced by S. If we take that force & divide by q then we obtain the "ELECTRIC FIELD" due to S at the point in question. For this to make sense we must insist q is small so S stays stuck in its static configuration. In other words, S acts on q through its electric field, but q does not act on S in any measureable way because q is so small. We use the same sort of idea with gravity, we take the EARTH as immovable because

Mass person << Mass of EARTH.

ELECTRIC FIELD CONTINUED

So here's a picture of what I said



$\vec{r}_j$ = position vector of j^{th} source charge q_j

$\vec{r} = \langle x, y, z \rangle$ = position of test charge. The location we wish to calculate $\vec{E}$

The vector from q_j to q is $\vec{\mathscr{r}}_j = \vec{r} - \vec{r}_j$.

The unit-vector $\hat{\mathscr{r}}_j$ points from q_j to q.

To find total field, just add them up,

$$\boxed{\vec{E}(\vec{r}) = \sum_{j=1}^{N} \frac{k q_j}{\mathscr{r}_j^2} \hat{\mathscr{r}}_j}$$

this is Eq: 19.6 in text. (pg 612) But, I make dependence on $P = \vec{r}$ explicit.

Aternatively,

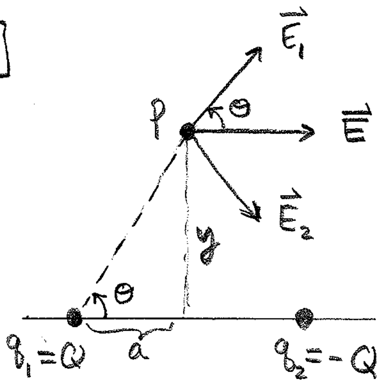
$$\boxed{\vec{E}(\vec{r}) = \sum_{j=1}^{N} \frac{k q_j}{\mathscr{r}_j^3} \vec{\mathscr{r}}_j}$$

If S is replaced with a continuous smeared out distribution of charge then we find $\sum \longrightarrow \int$. More on that later.

DIPOLE FIELD

(EXAMPLE 19.3)

E13



Place $q_1 = Q$ at $(-a, 0)$ and $q_2 = -Q$ at $(a, 0)$ and calculate the $\vec{E}$-field at $(0, y)$.

The distance from q_1 to $P = (0, y)$ is $\sqrt{a^2+y^2}$. The same is true for q_2. Notice that $(\vec{E}_1)_y$ cancels $(\vec{E}_2)_y$. We need only calculate the x-component,

Note $(\vec{E}_1)_x = E_1 \cos\theta = \dfrac{kQ}{a^2+y^2}\cos\theta = \dfrac{kQ}{a^2+y^2}\dfrac{a}{\sqrt{a^2+y^2}}$

Likewise $(\vec{E}_2)_x = E_2 \cos\theta = \dfrac{kQ}{a^2+y^2}\cos\theta = \dfrac{kQ}{(\sqrt{a^2+y^2})^3}$

$\therefore$ $\boxed{\vec{E} = \dfrac{kQ}{(a^2+y^2)^{3/2}}\,\hat{\imath} \quad \text{for any point } P = (0,y) \text{ where } y > 0}$

The argument above is the one given by text. I would use the formulation on 16 to find the same and more with ease;

$$\vec{r}_1 = \langle -a, 0\rangle \qquad \vec{\mathscr{r}}_1 = \vec{r} - \vec{r}_1 = \langle a, y\rangle$$

$$\vec{r}_2 = \langle a, 0\rangle \qquad \vec{\mathscr{r}}_2 = \vec{r} - \vec{r}_2 = \langle -a, y\rangle$$

$$\vec{r} = \langle 0, y\rangle \qquad \mathscr{r}_1 = \mathscr{r}_2 = \|\vec{\mathscr{r}}_1\| = \sqrt{a^2+y^2}.$$

$$\vec{E} = \frac{kQ}{\mathscr{r}_1^3}\vec{\mathscr{r}}_1 - \frac{kQ}{\mathscr{r}_2^3}\vec{\mathscr{r}}_2 = \frac{kQ}{\mathscr{r}_1^3}\langle a, y\rangle - \frac{kQ}{\mathscr{r}_2^3}\langle -a, y\rangle$$

cancel !

$$\boxed{\vec{E} = \frac{2kQa}{(\sqrt{a^2+y^2})^3}\langle 1, 0\rangle}$$

note $\mathscr{r}_1 = \mathscr{r}_2$.

Continuous Charge Distributions

$$\vec{E}(\vec{r}) = \underbrace{\frac{1}{4\pi\epsilon_0}\int \frac{dq}{r^2}\,\hat{r}}\; = \;\frac{1}{4\pi\epsilon_0}\iiint \frac{\rho(\vec{r})}{r^2}\,\hat{r}\,dV$$

this is interpreted as a scalar line integral, a integral over a surface, or sometimes a volume integral.

Note $\rho = \dfrac{dq}{dV}$ so $\rho\, dV = dq$

$\dfrac{\text{charge}}{\text{area}}$

$$\sigma = \frac{dq}{dA} \longrightarrow \vec{E}(\vec{r}) = k\iint \frac{\sigma(\vec{r})}{r^2}\,\hat{r}\,dA$$

$$\lambda = \frac{dq}{dl} \longrightarrow \vec{E}(\vec{r}) = k\int \frac{\lambda(\vec{r})}{r^2}\,\hat{r}\,dl$$

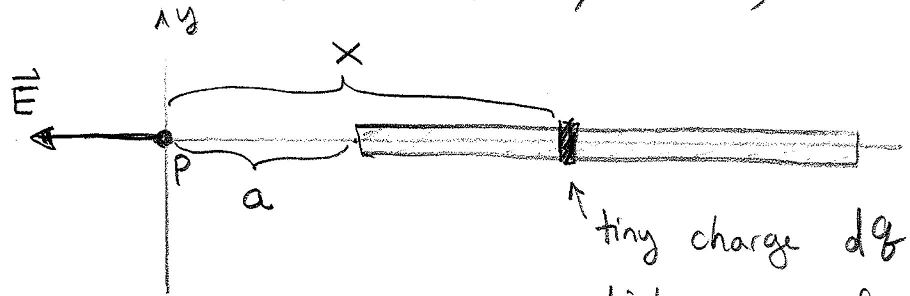
$\dfrac{\text{charge}}{\text{length}}$

Fortunately for you we usually have ρ, σ, λ as constants so these integrals are not so bad. Let's look at a couple easy examples.

(See 19.4 & 19.5)

Example 19.4: A charge Q is uniformly distributed along a rod of length l. Find the electric field at a point P a distance a from one end along the axis.

I'll follow the text's picture,



tiny charge $dq = \lambda dx$

distance x from P.

Contributes $d\vec{E} = \dfrac{-k\,dq}{x^2}\,\hat{i} = \dfrac{-k\lambda\,dx}{x^2}\,\hat{i}$

To find the total $\vec{E}$ field we add all the little $d\vec{E}$'s from each bit dq, this gives

$$\vec{E} = \int_a^{a+l} \frac{-k\lambda\,dx}{x^2}\,\hat{i}$$

$$= -k\lambda\hat{i} \int_a^{a+l} \frac{dx}{x^2}$$

$$= -k\lambda\hat{i}\left[\left.\frac{-1}{x}\right|_a^{a+l}\right]$$

$$= \frac{-kQ}{l}\left[\frac{-1}{a+l} + \frac{1}{a}\right]\hat{i}$$

$$= \frac{-kQ}{l}\left[\frac{-a + a + l}{a(a+l)}\right]\hat{i}$$

$$\boxed{\vec{E} = \frac{-kQ}{a(a+l)}\,\hat{i}}$$

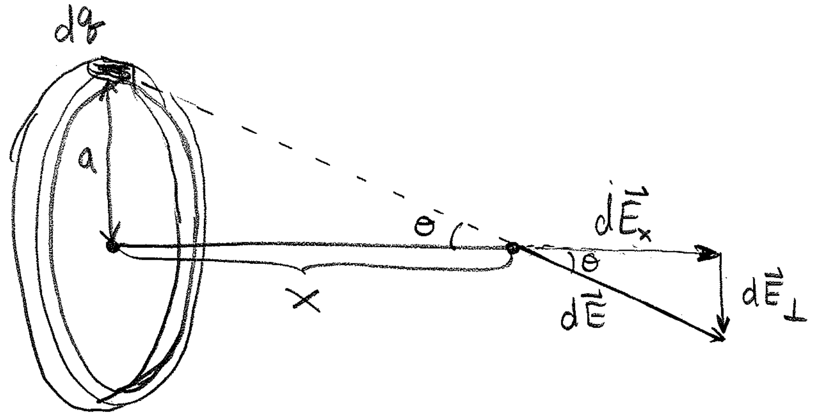
Note, as $a \longrightarrow \infty$ we find

$$\vec{E} \longrightarrow \frac{-kQ}{a^2}\,\hat{i}$$

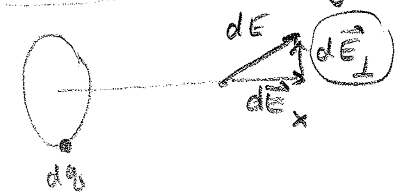
(Coulomb field)

E15

Example 19.5 A ring of radius a has a uniform positive charge Q. Calculate $\vec{E}$-field at a point along the axis of the ring a distance x from the center of the ring.



Note $d\vec{E}_\perp$ will cancel with other field from dq on opposite side of ring



We need to add up $d\vec{E}_x$ from all around the ring, $d\vec{E}_x = (\cos\theta \, dE)\,\hat{i}$

Note $dE = \dfrac{k\,dq}{(x^2+a^2)}$ & $\cos\theta = \dfrac{x}{\sqrt{x^2+a^2}}$

Notice $dq = \lambda\, ds = \dfrac{Q}{2\pi a}\, ds$ where ds = arclength subtended by dq.

Thus $d\vec{E}_x = \left(\dfrac{kQ\,ds}{2\pi a (x^2+a^2)}\right)\dfrac{x}{\sqrt{x^2+a^2}}\,\hat{i}$

$$\vec{E} = \int_{\text{ring}} \frac{kQx\,\hat{i}}{2\pi a (x^2+a^2)^{3/2}}\, ds = \frac{kQx\,\hat{i}}{2\pi a (x^2+a^2)^{3/2}} \underbrace{\int_{\text{ring}} ds}_{2\pi a}$$

$$\therefore \boxed{\vec{E} = \left[\frac{kQx}{\left(\sqrt{x^2+a^2}\right)^3}\right]\hat{i}}$$

Note as $x \longrightarrow \infty$ we find $E \longrightarrow \dfrac{kQx}{x^3} = \dfrac{kQ}{x^2}$

Coulomb Field!

Electric Field Lines

There are a number of ways to plot a vector field:

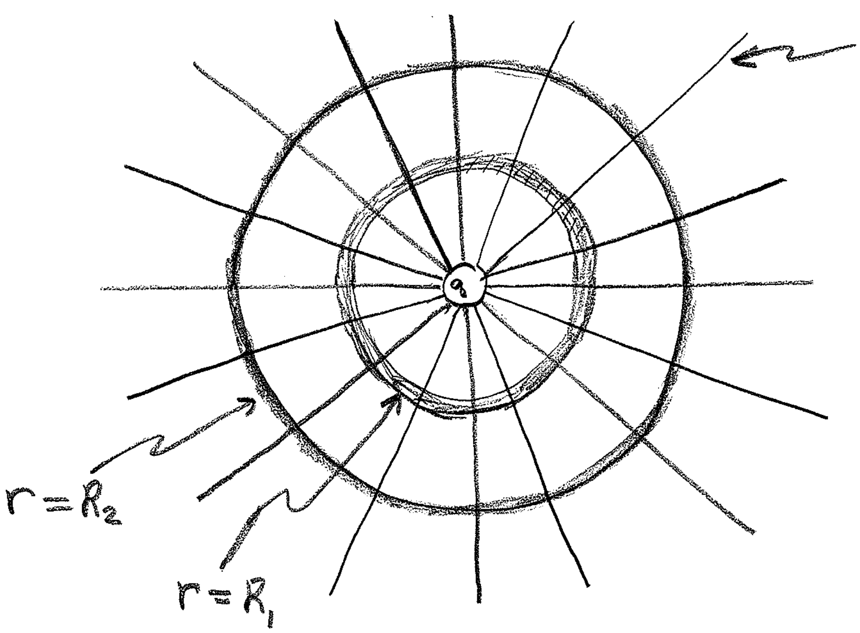
1. plot one sample vector per grid square and illustrate the magnitude by the length of the sample fields. (see pg 6, note I made vectors closer to $r=0$ bigger since $E \propto \frac{1}{r^2}$)

2. plot vectors of fixed magnitude. Plot more vectors per unit square if the field strength is larger.

3. Some combination of 1 and 2.

I'm not quite sure how Mathematica sets up the vector field plots we previously examined. It seems Mathematica is not attempting to show magnitude. Instead, its just trying to show the direction.

In physics it is probably wiser to use 2 since we will want to count vectors cutting through a surface. Also, 2 naturally works better with <u>field lines</u>. For example,



<u>field lines</u>: the tangents to these curves gives us vectors in $\vec{E}$ - direction.

Compare the field line density through $r=R_1$ and $r=R_2$. The line density is <u>proportional</u> to the <u>magnitude</u> of the field.

$$E_1 \propto \frac{N}{\underbrace{4\pi R_1^2}} \quad \& \quad E_2 \propto \frac{N}{4\pi R_2^2}$$

(area of spherical shell)

<u>Remark</u>: the eq$^{\underline{n}}$ $y \propto x$ means $\exists$ constant c such that $y = cx$

Continuing to analyze field lines for point charge in three-dimensions

The rules for drawing field lines given in the text are that:

1. lines flow <u>out</u> of positive charge
2. lines flow <u>into</u> negative charge
3. lines begin or end either at charge or $r=\infty$.

These rules lead me to draw the field lines on 21.

I claimed the <u>density</u> of field lines is proportional to the magnitude of the local field. Let's examine if this consistent with COULOMB'S LAW.

$$E_1 \propto \frac{N}{4\pi R_1^2} \quad \text{and} \quad E_2 \propto \frac{N}{4\pi R_2^2}$$

$$\Rightarrow E_1 = \frac{cN}{4\pi R_1^2} = \frac{q}{4\pi\epsilon_0 R_1^2} \quad \text{and} \quad E_2 = \frac{cN}{4\pi R_2^2} = \frac{q}{4\pi\epsilon_0 R_2^2}$$

We find the proportionality constant c is indeed constant for differing radii and in fact $c = q/N\epsilon_0$. If the "COULOMB" LAW WAS different then this 1, 2, 3 field line game would not work. However, the game does work and it leads us to Gauss' Law in it's most primitive form:

$$\left[\begin{array}{l}\text{\# of field lines}\\ \text{through a closed}\\ \text{surface}\end{array}\right] \propto \left[\begin{array}{l}\text{CHARGE}\\ \text{ENCLOSED}\end{array}\right]$$

The proportionality is governed by ϵ_0 and the choice of N for a given q. We <u>cannot</u> count all the field lines since there are ∞ly many. Statements about counting the # of lines are always relative to some choice of magnitude unit.

<u>Remark</u>: if the field spread out in a plane rather than all of $\mathbb{R}^3$ then the "Coulomb's Law" would need to look like $E = \frac{kq}{r}$ since the density of field lines would go like $\frac{1}{r}$; $\frac{N}{2\pi R_1} \propto E_1$ and $\frac{N}{2\pi R_2} \propto E_2$.

Coulomb's Law and Newton's Universal Law of Gravitation are the natural laws for a field which is isotropic and 3-dimensional.

Surface Integrals (math you should know by end of semester)

Given a vector field $\vec{F}$ and a surface S parametrized by $\vec{\Sigma} : D \longrightarrow S \subset \mathbb{R}^3$ we <u>define</u> the <u>flux of $\vec{F}$</u> through S by

$$\Phi_F = \int_S \vec{F} \cdot d\vec{A} = \iint_D \left(\vec{F}(\vec{\Sigma}(u,v)) \cdot \hat{N}\right) dS$$

Where $\hat{N} = \frac{1}{\|\vec{\Sigma}_u \times \vec{\Sigma}_v\|}\left(\vec{\Sigma}_u \times \vec{\Sigma}_v\right)$ is the unit-normal to the surface and $dS = \|\vec{\Sigma}_u \times \vec{\Sigma}_v\|\, du\, dv$ is the infinitesimal area element. Pragmatically the easier formula to use is simply,

$$\Phi_F = \int_S \vec{F} \cdot d\vec{A} = \iint_D \vec{F}(\Sigma(u,v)) \cdot (\Sigma_u \times \Sigma_v)\, du\, dv$$

If the surface S is <u>closed</u> then we insist that Σ is chosen to give S an outward pointing normal vector field $\vec{\Sigma}_u \times \vec{\Sigma}_v$ and we denote this by

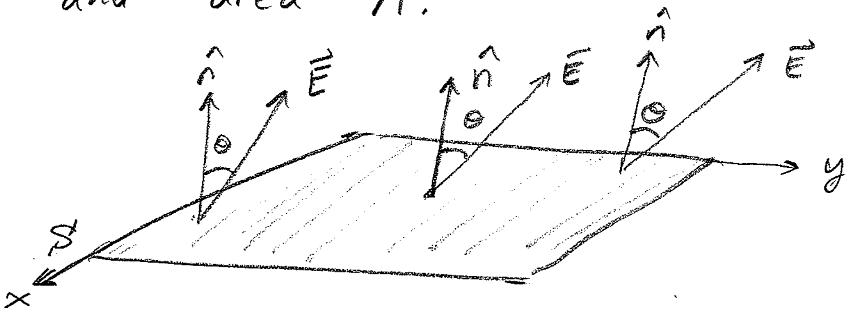
$$\Phi_F = \oint_S \vec{F} \cdot d\vec{A} \leftarrow \text{flux cutting through } S.$$

Your text roughly motivates the def$^{\underline{n}}$ of the surface integral, but you'll notice the mathematics is absent. In a physics' ~~text~~ the notation $\oint \vec{E} \cdot d\vec{A}$ is simply an invitation to mimick a standard example. (which we shall do) However, there is a careful def$^{\underline{n}}$ to give and this is why I have offered it. I'll be sprinkling a fair amount of calculus III here and there. It may be unfair to expect you understand it at this juncture but by the end of the semester since Math 232 is a co-requisite some question should be reasonable. Anyway, calculus III is the <u>best</u> way to understand the physical laws in this course, whether or not I go beyond the text example-wise.

<u>Remark:</u> as much as I would love to unravel bit by bit each flux-integral via the explicit concrete mathematics of parametrizations I will resist and instead use the pseudo-mathematical symmetry arguments that are envogue for this course. Details upon details can be found in my calculus III notes.

(In other words, in the pages that follow I use calculation by authority and tradition.)

E16 Constant vector field, plane with normal $\hat{n}$ and area A.

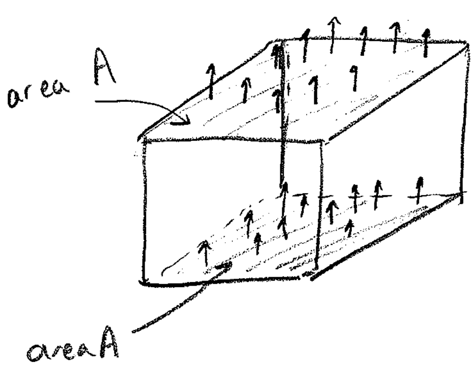


$$\Phi_E = \int_S \vec{E}\cdot d\vec{A} = \iint_S (\vec{E}\cdot\hat{n})\,dx\,dy = (\vec{E}\cdot\hat{n})\underbrace{\iint dx\,dy}_{\text{AREA } A}$$

$$\therefore \quad \boxed{\Phi_E = EA\cos\theta} \quad \text{(easiest case)}$$

Note, if $\theta = 0$ then $\Phi_E = EA$ whereas $\theta = 90° \Rightarrow \Phi_E = 0$. If $\theta = 180°$ then $\Phi_E = -EA$, means field opposite normal $\hat{n}$.

E17 Closed Cube and constant $\vec{E}$, keep it simple $\vec{E} = E_o\hat{k}$



$$\Phi_{CUBE} = \oint_{CUBE} \vec{E}\cdot d\vec{A}$$

$$= \underbrace{\int_{BOTTOM} \vec{E}\cdot d\vec{A}}_{d\vec{A} = -\hat{k}\,dA} + \underbrace{\int_{TOP} \vec{E}\cdot d\vec{A}}_{d\vec{A} = \hat{k}\,dA} + \underbrace{\int_{\substack{VERTICAL\\FACES}} \vec{E}\cdot d\vec{A}}_{d\vec{A} \perp \vec{E}\ \Rightarrow \text{zero.}}$$

$$= -E_oA + E_oA$$

$= 0$: the flux E_oA out the top face is cancelled by the flux $-E_oA$ from the field entering the base of the cube.

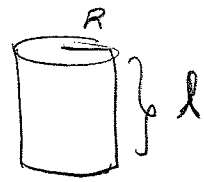
Note: all closed surfaces are by convention given an outward-pointing normal in this course. This insures the def$^{\underline{n}}$ of $\Phi_E = \oint_S \vec{E}\cdot d\vec{A}$ is unambiguous once S is specified.

$\boxed{E18}$ If $\vec{E} = E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k}$ and we again consider the flux through a cube we'll find zero net-flux since,

$$\Phi_E = \oint \vec{E} \cdot d\vec{A}$$

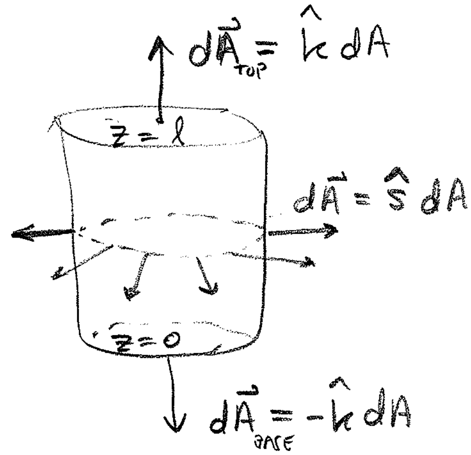
$$= \underbrace{\oint E_{0x}\hat{i} \cdot d\vec{A} + \oint E_{0y}\hat{j} \cdot d\vec{A}}_{\text{same argument as } \boxed{E17} \text{ just switch } x \longleftrightarrow z \text{ or } y \longleftrightarrow z.} + \underbrace{\oint E_{0z}\hat{k} \cdot d\vec{A}}_{\text{see } \boxed{E17}}$$

$\boxed{E19}$ Suppose $\vec{E}(s,\theta,z) = E(s)\,\hat{s}$ where $\hat{s}$ is defined $\hat{s} = \dfrac{\nabla s}{|\nabla s|}$ and $s = \sqrt{x^2+y^2}$ where $x = s\cos\varphi$, $y = s\sin\varphi$, $z = z$ are cylindrical coordinates. The flux through a cylinder is easy to calculate: $C = \{(x,y,z) \mid s \leq R,\ 0 \leq z \leq l\}$



$$\Phi_E = \oint_C \vec{E} \cdot d\vec{A}$$

$$= \underset{\substack{z=0 \\ (\text{BASE})}}{\int} \vec{E} \cdot d\vec{A}_{BASE} + \underset{\substack{s=R \\ (\text{CURVED SIDES})}}{\int} \vec{E} \cdot d\vec{A} + \underset{\substack{z=l \\ (\text{TOP})}}{\int} \vec{E} \cdot d\vec{A}_{TOP}$$



$$= \int_{s=R} E(s)\,\hat{s} \cdot \hat{s}\,dA, \quad \left(\text{noting } d\vec{A}_{TOP} \text{ \& } d\vec{A}_{BASE} \text{ are } \perp \text{ to } \vec{E}.\right)$$

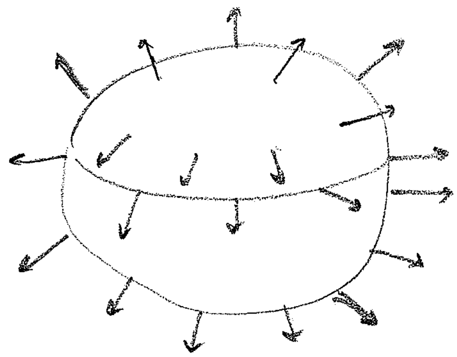
$$= E(R) \int_{s=R} dA$$

$$= E(R) \cdot (2\pi R)\,l$$

It is important that I <u>knew</u> $\vec{E} \parallel d\vec{A}$ for the curved sides. If $\vec{E}$ lacks that symmetry you actually have to flesh out the details of 23

$\boxed{E20}$ Suppose $x = r\cos\varphi\sin\theta$, $y = r\sin\varphi\sin\theta$, $z = r\cos\theta$ where $0 \leq \varphi \leq 2\pi$ and $0 \leq \theta \leq \pi$ (PHYSICS' CONVENTIONS)

then $r = \sqrt{x^2+y^2+z^2}$ and $\hat{r} = \dfrac{\nabla r}{\|\nabla r\|} = \langle \cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta \rangle$



Anyway, $\hat{r}$ is the spherically-symmetric vector field of unit-length. It is well-defined for $r \neq 0$. We used it to define the Coulomb field. $\vec{E} = \dfrac{kq}{r^2}\hat{r}$.

Suppose we have $\vec{E}(r,\varphi,\theta) = E(r)\hat{r}$. This means $\vec{E}$ depends only on the spherical radius. Let S_R be the sphere of radius R centered at the origin

$$\Phi_E = \oint_{S_R} \vec{E} \cdot d\vec{A} = \int_{r=R} (E(r)\hat{r}) \cdot (\hat{r}\, dA) = E(R) \int_{r=R} dA$$

$$\therefore \boxed{\Phi_E = E(R) \cdot 4\pi R^2}$$

Again, if $\vec{E}$ was not spherically symmetric this shortcut calculation would be replaced with a careful calculation based on the def^{n} on page 23

<u>Remark</u>: as far as I recall, Freshman physics texts are incapable of handling geometries other than the plane, cylinder, or sphere. Therefore, this bank of examples should suffice for our future work. Other geometries involve indirect <u>physical</u> reasoning. You should not find yourself using the def^{n} on 23 to "do" problems. (well, unless I write a problem for precisely that purpose)

Mathematical Digression: SURFACE AREA OF SPHERE

$$\Sigma(\theta,\varphi) = \langle R\cos\varphi\sin\theta,\ R\sin\varphi\sin\theta,\ R\cos\theta\rangle$$

for $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$ parametrizes the sphere $r = R$ where $r^2 = x^2+y^2+z^2$. Calculate

$$\frac{\partial \Sigma}{\partial \theta} = \Sigma_\theta = \langle R\cos\varphi\cos\theta,\ R\sin\varphi\cos\theta,\ -R\sin\theta\rangle$$

$$\frac{\partial \Sigma}{\partial \varphi} = \Sigma_\varphi = \langle -R\sin\varphi\sin\theta,\ R\cos\varphi\sin\theta,\ 0\rangle$$

After a short calculation,

$$\Sigma_\theta \times \Sigma_\varphi = \langle R^2\cos\varphi\sin^2\theta,\ R^2\sin\varphi\sin^2\theta,\ R^2\sin\theta\cos\theta\rangle$$

$$\therefore\ \Sigma_\theta \times \Sigma_\varphi = R^2\sin\theta\,\underbrace{\langle \cos\varphi\sin\theta,\ \sin\varphi\sin\theta,\ \cos\theta\rangle}_{\hat{r}}$$

(not surprising!)

$$\therefore\ \underline{\Sigma_\theta \times \Sigma_\varphi = R^2\sin\theta\,\hat{r}}.$$

$$\Rightarrow\ \|\Sigma_\theta \times \Sigma_\varphi\| = R^2\sin\theta$$

(note $\sin\theta \ge 0$ for $\theta \in [0,\pi]$)

$$\text{Area} = \int_{r=R} dA = \int_0^{2\pi}\int_0^{\pi} R^2\sin\theta\, d\theta\, d\varphi$$

$$= R^2\left(-\cos\theta\Big|_0^{\pi}\right)\left(\varphi\Big|_0^{2\pi}\right)$$

$$= R^2\,(2)\,(2\pi)$$

$$= \underline{4\pi R^2}.$$

Note then $\text{Vol Sphere} = \displaystyle\int_0^R 4\pi r^2\,dr = \frac{4\pi r^3}{3}\Big|_0^R = \frac{4}{3}\pi R^3.$

<u>Remark</u>: in my experience, calculus III students struggle most with the trouble of finding the parametrization for a given S. Constructing S's parametrization Σ is an art in general.