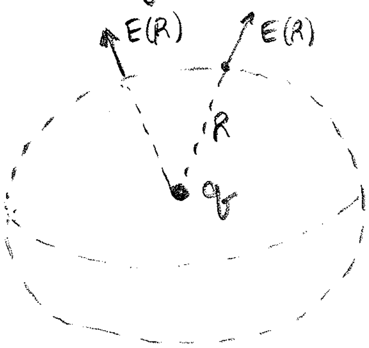
Gauss' Law

The net electric flux (Φ_E) through a closed surface S is given by the charge enclosed divided by ϵ_0;

$$\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc by } S}}{\epsilon_0}$$

<u>Remark:</u> Gauss' or the divergence Thm states $\int \vec{E} \cdot d\vec{A} = \iiint (\nabla \cdot \vec{E})\, dV$ hence $\oint_{\partial B} \vec{E} \cdot d\vec{A} = \iiint_B (\nabla \cdot \vec{E})\, dV = \frac{1}{\epsilon_0} Q_{enc} = \frac{1}{\epsilon_0} \iiint_B \rho\, dV$ and since this holds for arbitrary regions B it follows that $\nabla \cdot \vec{E} = \rho/\epsilon_0$. This is the <u>local</u> statement of Gauss' Law. (it's the real Gauss' Law imho).

$\boxed{\text{E21}}$ Consider a single point charge at $r = 0$. Imagine surrounding it with a spherical surface S_R at $r = R$.



Note that $\vec{E}$ from q must be spherically symmetric since no particular θ, ϕ are preferred by the geometry of the charge distribution $\therefore\ \vec{E} = E(r)\,\hat{r}$

Apply Gauss' Law with this symmetry in mind,

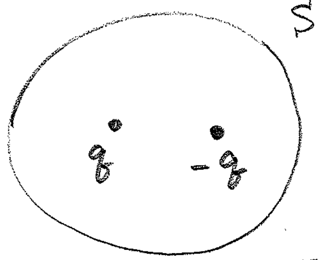
$$\Phi_E = \frac{Q_{enc}}{\epsilon_0} \longrightarrow \underbrace{\oint_{S_R} \vec{E} \cdot d\vec{A}}_{} = \frac{q}{\epsilon_0}$$

$$\underbrace{E(R)}_{\substack{\text{magnitude of} \\ \vec{E} \text{ at } r = R.}} \cdot 4\pi R^2 = \frac{q}{\epsilon_0}$$

Hence, $E(R) = \dfrac{q}{4\pi\epsilon_0 R^2} \Rightarrow \underline{\vec{E}(\vec{r}) = \dfrac{q}{4\pi\epsilon_0} \dfrac{\hat{r}}{r^2}}$.

put $R = r$ again and recall the direction

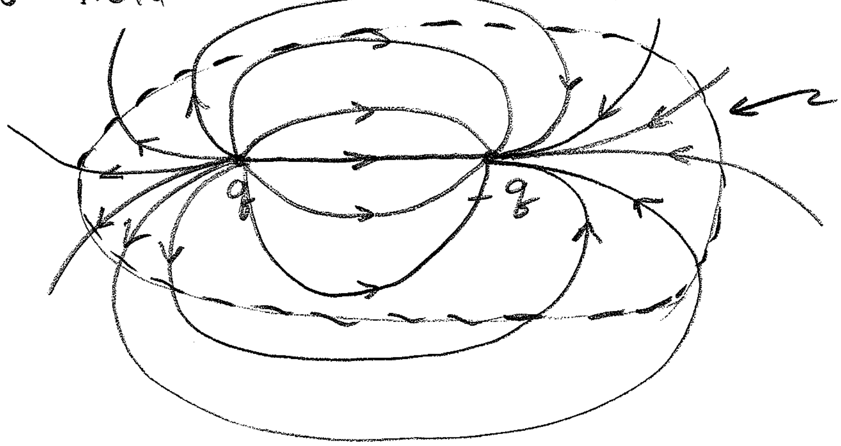
E22 Calculate net-flux through S pictured below



$S \leftarrow$ a closed surface which encloses q, $-q$ and no other charge.

Sol^n/ $\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = \dfrac{Q_{enc}}{\epsilon_o} = \dfrac{q - q}{\epsilon_o} = \boxed{0.}$

We can understand this by sketching a few field lines for this <u>dipole</u> distribution

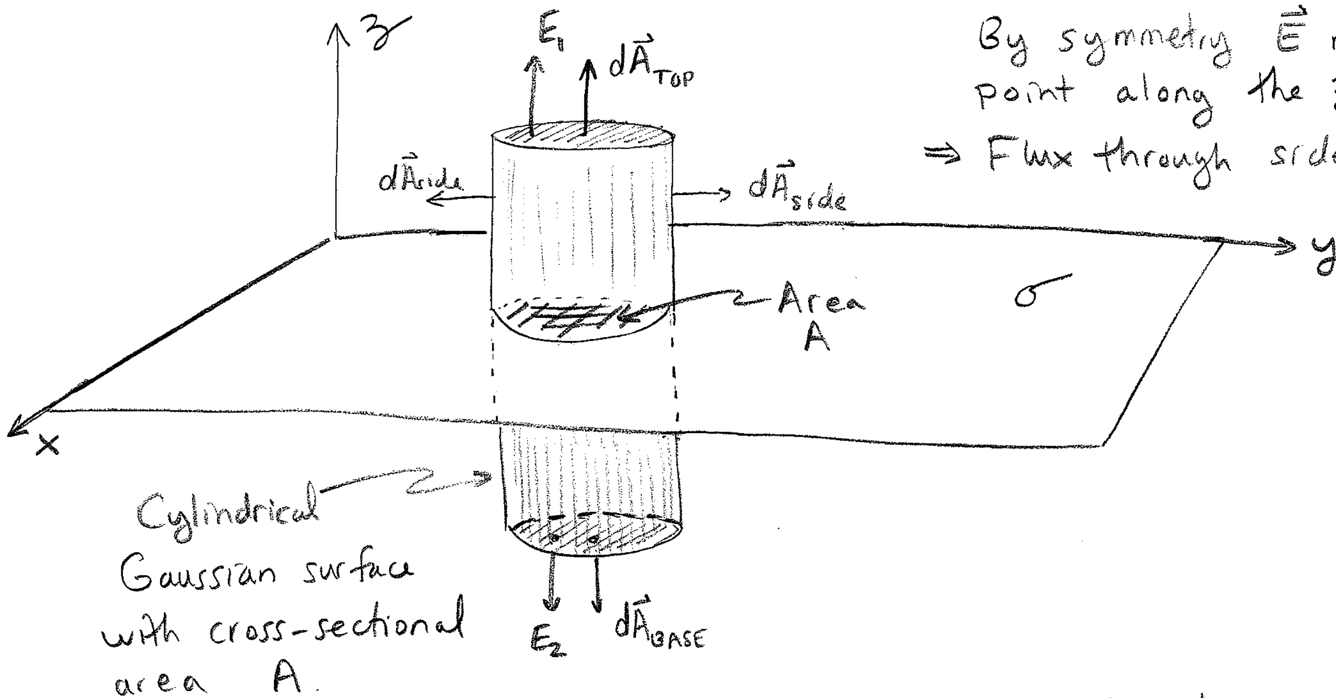


$S \leftarrow$ is mathematical construct. A "Gaussian surface"

<u>Remark</u>: If the charge distribution is not especially symmetrical then Gauss' Law tends not to help finding the explicit form of $\vec{E}$. We found the dipole field previously, it's not terribly complicated. On the other hand, S was pretty-much random here so explicit computation of $\Phi_E = \oint_S \vec{E} \cdot d\vec{A}$ is very complicated.

(Behold the beauty of Gauss' Th^{m}.)

$\boxed{\text{E23}}$ Find the electric field due to an infinite plane of charge with density σ.



By symmetry $\vec{E}$ must point along the z-direction.
$\Rightarrow$ Flux through sides zero.

<u>Apply Gauss' Law</u>: note $Q_{enc} = \sigma A$ hence

$$\Phi_E = \frac{Q_{enc}}{\epsilon_0} \implies \oint_S \vec{E}\cdot d\vec{A} = \frac{\sigma A}{\epsilon_0}$$

$$\int_{S_{TOP}} \vec{E}\cdot d\vec{A} + \int_{S_{BASE}} \vec{E}\cdot d\vec{A} + \int_{SIDE} \vec{E}\cdot d\vec{A} = \underbrace{E_1 A + E_2 A}_{} = \frac{\sigma A}{\epsilon_0}$$

By symmetry $E_1 = E_2$.

$$\Rightarrow \boxed{E = \frac{\sigma}{\epsilon_0}}$$

The Electric Field from a plane of charge is σ/ϵ_0 and it points perpendicular to the plane (away from if $\sigma > 0$ and towards if $\sigma < 0$).

with coordinates as in picture

$$\boxed{\vec{E}(\vec{r}) = \begin{cases} \frac{\sigma}{\epsilon_0}\hat{k} & \text{if } z > 0 \\ \frac{-\sigma}{\epsilon_0}\hat{k} & \text{if } z < 0 \end{cases}}$$

ELECTROSTATIC EQUILIBRIUM IN CONDUCTORS

As the text discusses at length on pgs. 630-631. there are 4 important facts about charge in static equilibrium on a conductor,

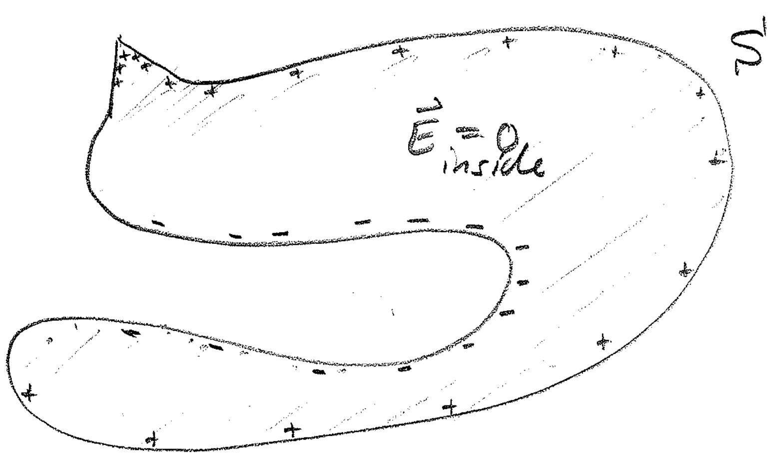
1.) $\vec{E} = 0$ inside a conductor.

2.) any net-charge is found on the boundary of the conductor

3.) $\vec{E} \perp$ surface of conductor, or $\vec{E}$ is parallel to outward normal of the conductor. Its magnitude is σ/ϵ_0 where σ is the <u>local</u> charge density $\frac{dQ}{dA}$

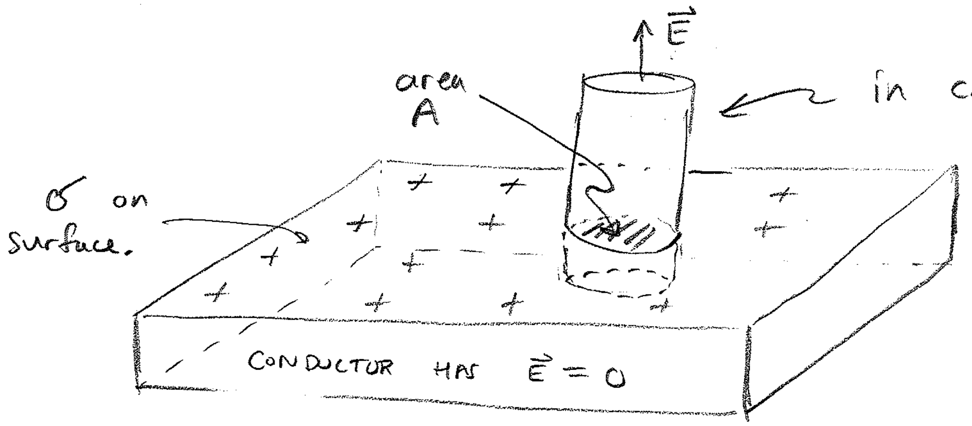
4.) On a lumpy conductor there is more charge near corners. Generally the larger the curvature the greater the σ.

Here's a picture of these



You'd need external charges to induce this charge distribution on S

$\boxed{E24}$ Consider conducting plane with following distribution of charge (pictured below)



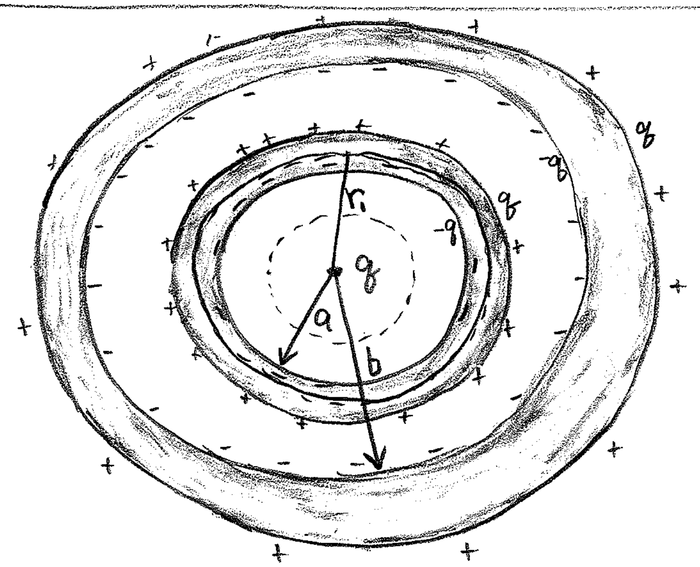
in contrast to $\boxed{E23}$ the BASE of the Gaussian cylinder has <u>zero</u> flux.

$$\Phi_E = \frac{\sigma A}{\epsilon_0} = \oint_{\text{cylinder}} \vec{E} \cdot dA = E \cdot A$$

: only top gives nonzero flux by symmetry & $\vec{E} = 0$ <u>inside</u> conductor.

$$\Rightarrow \boxed{\vec{E} = \frac{\sigma}{\epsilon_0} \hat{k} \quad \text{for points above the conductor}}$$

$\boxed{E25}$ Nested conducting spherical shells are placed around a charge of q. Find the charge on each surface and describe the $\vec{E}$ field everywhere except those surfaces



net-charge enclosed by interior of conductor must be zero. For example, for the sphere at $r = r_1$ we enclose q and the surface charge at $r = a$. Since $\Phi_E = 4\pi r_1^2 E_1$ by symmetry and $q_{enc} = q + \text{surface charge}$ we must have $\text{surface charge} = -q$, otherwise $\Rightarrow E_1 \neq 0$

E25 Continued

Since $E_1 = 0$ since $r = r_1$ is inside a conductor we must have $-q$ distributed over $r = a$.

But, since the conducting sphere at $r_1 = a$ was uncharged we must displace a charge of q to the other side of the conductor.

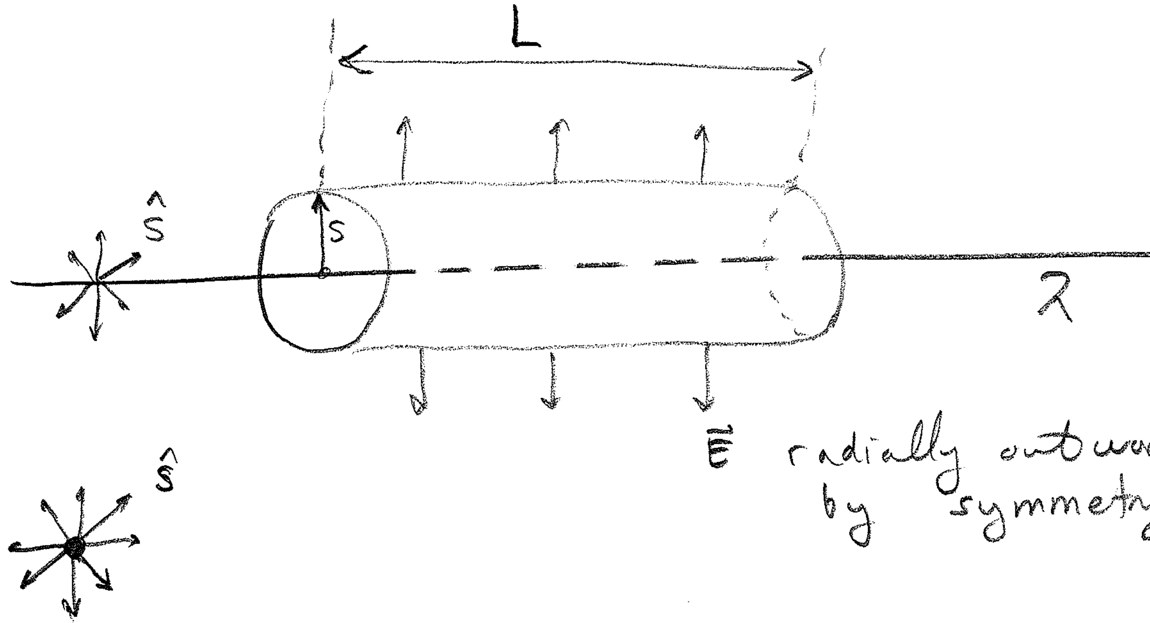
Likewise we induce $-q$ & q on the shell at $r = b$.

Now that we've figured out the charge distribution it's easy to find the $\vec{E}$-field through successive applications of Gauss' Law.

$$\vec{E}(r) = \begin{cases} \frac{kq}{r^2}\hat{r} & 0 < r < a \\ 0 & r = a \\ \frac{kq}{r^2}\hat{r} & a < r < b \\ \frac{kq}{r^2}\hat{r} & r > b \end{cases}$$

(assuming shells are very thin).

E26 Find $\vec{E}$-field from a very long line of charge with density λ = constant



$\vec{E}$ radially outward by symmetry.

$$\Phi_E = (L \cdot 2\pi s) E = \frac{Q_{enc}}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$$

$$\Rightarrow \quad E = \frac{\lambda}{2\pi\epsilon_0 s}$$

$$\Rightarrow \quad \boxed{\vec{E} = \frac{\lambda}{2\pi\epsilon_0} \frac{\hat{s}}{s}}$$

field points radially outward from the wire.