The Electric Potential

To begin we review a few results from mechanics. In particular we said $\vec{F}$ was a <u>conservative force</u> iff there was a potential energy function U such that $\vec{F} = -\nabla U = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$.
If the motion of a particle is due to $\vec{F}$ then we can prove the total energy $E = \frac{1}{2}m\vec{v}\cdot\vec{v} + U$ is conserved along the eq$^{\underline{n}}$'s of motion.

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2}m\vec{v}\cdot\vec{v} + U\right]$$

$$= \frac{1}{2}m\left(\frac{d\vec{v}}{dt}\cdot\vec{v} + \vec{v}\cdot\frac{d\vec{v}}{dt}\right) + \frac{dU}{dt}$$

$$= \underbrace{m\,\vec{v}\cdot\frac{d\vec{v}}{dt}}_{} + \underbrace{\frac{d\vec{r}}{dt}\cdot\nabla U}_{}$$

notice $\vec{v} = \frac{d\vec{r}}{dt}$ and

chain-rule for $U = U(\vec{r})$

$$\frac{dU(x,y,z)}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt}$$

$$= \langle \partial_x U, \partial_y U, \partial_z U\rangle \cdot \langle \dot{x}, \dot{y}, \dot{z}\rangle$$

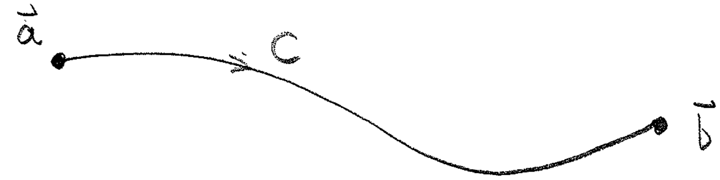
$$= \nabla U \cdot \frac{d\vec{r}}{dt}.$$

$$= \left(m\frac{d\vec{v}}{dt} + \nabla U\right)\cdot\frac{d\vec{r}}{dt}$$

$$= (m\vec{a} - \vec{F})\cdot\vec{v}$$

$$= 0 \quad \text{provided } \vec{F} = m\vec{a}$$ (Eq$^{\underline{n}}$'s of Motion!)

I showed in lecture that the line-integral of the force calculates the $-\Delta U$. The work done by $\vec{F}$ along some curve C is <u>minus</u> the change in U along the same path



$$W_c = \int_c \vec{F}\cdot d\vec{l} = \int_c -(\nabla U)\cdot d\vec{l} \qquad (\overbrace{\vec{F} = -\nabla U}^{\text{conservative force.}})$$

$$= -\int_c (\nabla U)\cdot d\vec{l}$$

$$= -(U(\vec{b}) - U(\vec{a}))$$

$$= \underline{-\,\Delta PE \text{ from } \vec{a} \text{ to } \vec{b}.}$$

It is generally true that the work done by a force along C will be equal to the net-change in the Kinetic Energy $KE = T = \frac{1}{2}m\vec{v}\cdot\vec{v}$. We can show

$$W_c = \int_c \vec{F}\cdot d\vec{l} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = \underline{\Delta KE}$$

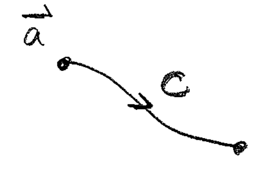
Thus $\Delta KE = -\Delta PE$ <u>if</u> $\vec{F} = -\nabla U$ for some potential energy fnct. U. This gives us <u>energy conservation</u>

$$\boxed{E_a = \frac{1}{2}mv_a^2 + U(\vec{a}) = \frac{1}{2}mv_b^2 + U(\vec{b}) = E_b}$$

(not generally true if $\vec{F} \neq -\nabla U$ for some $U = U(x,y,z)$).

Changes in PE

To calculate the change in PE from $\vec{a} \longrightarrow \vec{b}$ we may simply calculate,



$$\boxed{\Delta PE \text{ from } \vec{a} \text{ to } \vec{b} = -\int_C \vec{F} \cdot d\vec{l}}$$

E27 Suppose $\vec{F} = F_o \hat{i}$ and let C be path from $\underbrace{(0,0) \text{ to } (3,3)}$ in meters.

C parametrized by $\vec{r}(t) = (3t, 3t)$ for $0 \leq t \leq 1$

$$\begin{aligned}
\Delta PE_{(0,0) \to (3,3)} &= -\int_0^1 (F_o \hat{i}) \cdot \left(\frac{d\vec{r}}{dt}\right) dt \\
&= -\int_0^1 F_o \hat{i} \cdot \langle 3, 3 \rangle \, dt \\
&= -\int_0^1 3F_o \, dt \\
&= -3F_o t \Big|_0^1 \\
&= -3F_o. \\
&= -F_o \Delta X
\end{aligned}$$

$(\Delta X = 3 - 0$ change in $x)$

E28 To find change in PE from $(0,0) \longrightarrow (3,3)$ it's easier to just notice $U = -F_o x$ has

$$\nabla U = \left\langle \frac{\partial}{\partial x}(-F_o x), \frac{\partial}{\partial y}(-F_o x) \right\rangle = \langle -F_o, 0 \rangle = -F_o \hat{i}$$

Thus $\vec{F} = -\nabla U = F_o \hat{i}$. Then

$$\begin{aligned}
\Delta PE_{(0,0) \to (3,3)} &= U(3,3) - U(0,0) \\
&= -F_o(3) + F_o(0) \\
&= -3F_o.
\end{aligned}$$

(these examples are silly w/o what follows)

Continuing to think about $\boxed{E27}$ & $\boxed{E28}$

$\boxed{E29}$ Suppose a particle moves as a consequence of the <u>net</u> force $\vec{F} = F_0 \hat{i}$ from $(0,0)$ to $(3,3)$. If the particle is initially at rest then how fast is it moving at $(3,3)$? We can use <u>energy</u> <u>conservation</u> to assist us since we just showed in $\boxed{E28}$ this force is conservative.

$$E_i = E_f$$

$$\tfrac{1}{2}m\underbrace{V_i^2}_{0} + \underbrace{U(0,0)}_{0} = \tfrac{1}{2}mV_f^2 + U(3,3)$$

$$\Rightarrow \tfrac{1}{2}mV_f^2 = -U(3,3) = -(-F_0(3)) = 3F_0$$

$$\Rightarrow \underline{V_f = \sqrt{\frac{6F_0}{m}}}.$$

(units omitted on 3 since I don't want to confuse m = mass with m = meters)

<u>Remark</u>: given a particular vector field $\vec{F}$ there are many ways to check if it is conservative but finding $U = U(x,y,z)$ such that $\vec{F} = -\nabla U$ is probably the most useful. Once we find $U = PE$ then energy conservation helps us analyze motion w/o even solving $\vec{F} = m\vec{a}$ directly. See pg. 420 of my calculus III notes for more on why TFAE

(1.) $\vec{F}$ conservative

(2.) $\exists f$ s.t. $\vec{F} = \nabla f$

(3.) $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$ $\forall$ paths C_1 & C_2 with same terminal pts.

(4.) dom$(\vec{F})$ <u>simply connected</u> and $\nabla \times \vec{F} = 0$

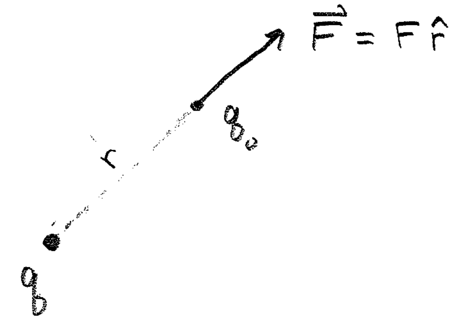
(5.) $\oint_C \vec{F}\cdot dr = 0$ for all loops C.

(BONUS COMMENT)

Applying conservative force theory to electric force

The Coulomb force due to q at origin acting on q_0 at $\vec{r}$ is given by

$$\vec{F} = \frac{k q q_0}{r^2}\hat{r}$$



Using multivariate calculus and spherical coordinates one can show

$$U = \frac{k q q_0}{r}$$

$$\Rightarrow \nabla U = \frac{-k q q_0}{r^2}\hat{r} \Rightarrow \underline{-\nabla U = \frac{k q q_0}{r^2}\hat{r} = \vec{F}}.$$

Thus the Coulomb Force is conservative **and** it has PE function

$$\boxed{U(r) = \frac{k q q_0}{r}}$$

(can add constant here, there is freedom)

Obviously this depends on both q & q_0. However, we would like to drop the dependence on q_0 and use a concept similar to the electric field. Recall $\vec{E} = \vec{F}/q_0$ thus define the PE per unit charge q_0 as follows:

$$\boxed{V(r) = \frac{k q}{r}}$$

(can add constant $\exists$ freedom!)

The formulas for PE and "electric potential" V were given with the assumption that $U(r=\infty)$ and $V(r=\infty)$ should be zero. This is merely a convention. Generally we may define (for the point charge q at $\vec{r}=0$)

$$\boxed{\begin{aligned} U(\vec{r}) &= \frac{kqq_0}{r} + U_0 \\ V(\vec{r}) &= \frac{kq}{r} + V_0 \end{aligned}}$$

For some problems we'll need to make $V_0 \neq 0$. However, the ΔV is independent of the choice of V_0 since

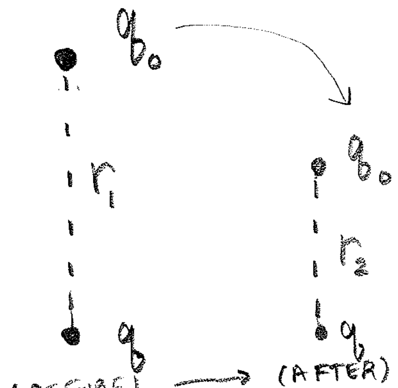
$$\Delta V = V(\vec{r}_2) - V(\vec{r}_1) = \left(\frac{kq}{r_2} + V_0\right) - \left(\frac{kq}{r_1} + V_0\right)$$

$$\boxed{\Delta V = kq\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}$$

To find the corresponding ΔPE for q_0 as it goes from $\vec{r}_1 \longrightarrow \vec{r}_2$ under the net-force $\vec{F}_{coulomb}$ we simply multiply by q_0

$$\boxed{\Delta U = \Delta PE = q_0 \Delta V = kqq_0\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}$$

$\boxed{\text{E30}}$ Study energy for two charges



$$\Delta PE = k\underbrace{q q_0}_{+/-}\underbrace{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}_{\text{positive}}$$

notice $r_1 > r_2$
hence $\frac{1}{r_2} > \frac{1}{r_1}$
$\Rightarrow \frac{1}{r_2} - \frac{1}{r_1} > 0$

CASES

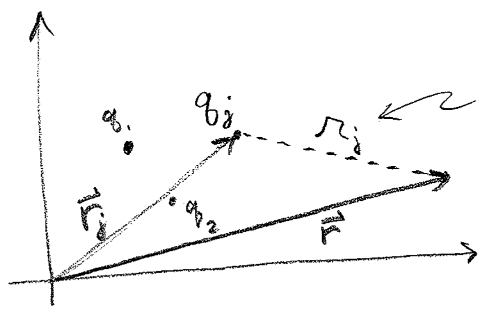
1.) if $q q_0 > 0$ then $\Delta PE > 0$
$\Rightarrow \Delta KE < 0$ thus q_0 must slow down.
(if $v = 0$ at r_1 then this motion is not physically reasonable)
} repulsive $\vec{F}$

2.) if $q q_0 < 0$ then $\Delta PE < 0$
$\Rightarrow \Delta KE > 0$ thus q_0 must speed up. (no unphysical case)
} attractive $\vec{F}$

As I elaborated in some depth in the email we ought only make analogies to case (2.) in comparing to gravity since gravity is always attractive. The spring analogy works both ways though.
(1.) like compressing. (2.) like elongating spring.

Notice, I've only stated V for point charge. If we have many and take $V=0$ at ∞ then the net-potential is simply

$$V(r) = \sum_{j=1}^{n} \frac{k q_j}{\mathscr{r}_j}$$



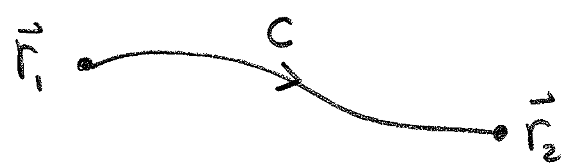
$\mathscr{r}_j = |\vec{r} - \vec{r}_j|.$

If continuous then use $V(r) = \int \frac{k\,dq}{\mathscr{r}}$

over the distribution
may be line, surface or area integral.

- Suppose we are given the $\vec{E}$-field then the potential difference from $\vec{r}_1$ to $\vec{r}_2$ is calculated by

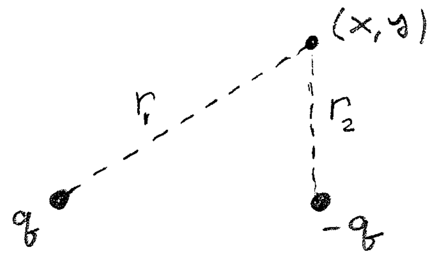
$$V_2 - V_1 = \Delta V = -\int_C \vec{E} \cdot d\vec{l}$$



(since $\vec{F} = q\vec{E}$ conservative <u>any</u> path C gives same result.)

E31 potential for a dipole with q at the point $(-d, 0)$ and $-q$ at $(d, 0)$ where the length of this dipole is $2d$

I'm thinking of $q > 0$ however the mathematics allows $q < 0$ in the formulas below.

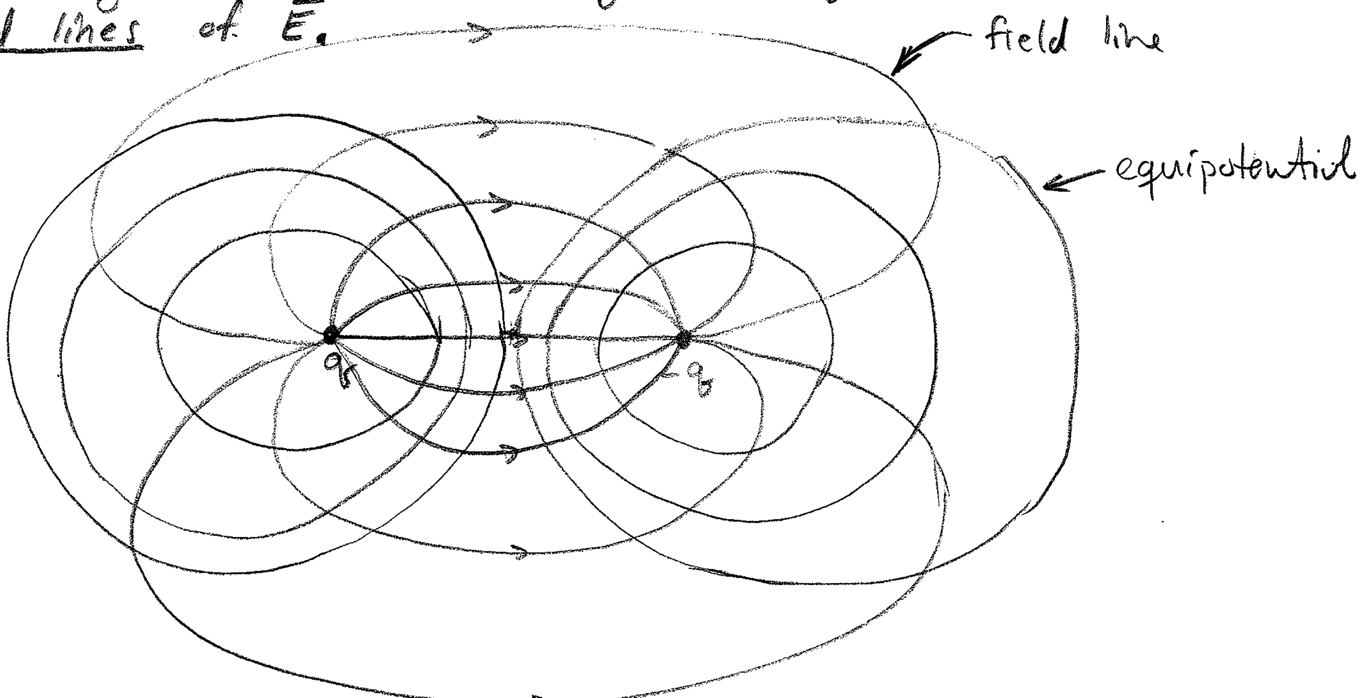


$$\vec{r}_1 = \langle x+d, y \rangle$$
$$\vec{r}_2 = \langle x-d, y \rangle$$

$$V = \frac{kq}{r_1} - \frac{kq}{r_2} \quad (\text{choosing } V(\infty) = 0)$$

$$\Rightarrow \boxed{V(x,y) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x+d)^2+y^2}} - \frac{1}{\sqrt{(x-d)^2+y^2}}\right]}$$

You can calculate the electric field $\vec{E}$ due to this dipole field by the formula $\vec{E} = -\nabla V$. Geometrically, this makes lines of constant V (equipotentials) have tangents which are <u>orthogonal trajectories</u> to the <u>field lines</u> of $\vec{E}$.



(If picture drawn carefully then they always meet at right angles.)

E31 Continued

The explicit computation of $\vec{E}(x,y)$ from $V(x,y)$ is slightly unpleasant, but I'll do it anyway just to show you how.

$$\vec{E} = -\nabla V = \left\langle -\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y} \right\rangle$$

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left[\frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x+d)^2+y^2}} - \frac{1}{\sqrt{(x-d)^2+y^2}}\right]\right] \quad : \quad \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{u}}\right] = \frac{1}{u^{3/2}}\frac{\partial u}{\partial x}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{-(x+d)}{\left[\sqrt{(x+d)^2+y^2}\right]^3} + \frac{x-d}{\left[\sqrt{(x-d)^2+y^2}\right]^3}\right]$$

$$= \frac{-q}{4\pi\epsilon_0}\left[\frac{x+d}{r_1^3} - \frac{x-d}{r_2^3}\right].$$

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}\left[\frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x+d)^2+y^2}} - \frac{1}{\sqrt{(x-d)^2+y^2}}\right]\right]$$

$$= \frac{-q}{4\pi\epsilon_0}\left[\frac{y}{\left[\sqrt{(x+d)^2+y^2}\right]^3} - \frac{y}{\left[\sqrt{(x-d)^2+y^2}\right]^3}\right]$$

$$= \frac{-q}{4\pi\epsilon_0}\left[\frac{y}{r_1^3} - \frac{y}{r_2^3}\right]$$

Hence, $\vec{E} = \left\langle -\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}\right\rangle = \frac{q}{4\pi\epsilon_0}\left\langle \frac{x+d}{r_1^3} - \frac{x-d}{r_2^3}, \frac{y}{r_1^3} - \frac{y}{r_2^3}\right\rangle$

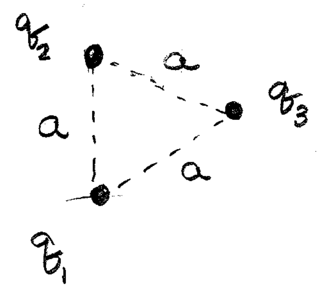
Which we can "simplify" to $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{r_1^3}\langle x+d, y\rangle - \frac{q}{4\pi\epsilon_0}\frac{1}{r_2^3}\langle x-d, y\rangle$.

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}_1}{r_1^3} - \frac{q}{4\pi\epsilon_0}\frac{\vec{r}_2}{r_2^3}.$$

It's straightforward, it just a little differentiation.

Remark: look at Example 20.4 on 651-652 for a special case of my $\boxed{E31}$.

E32 What is the total work required to put 3 point charges (pictured below) in place. Assume they are initially very very far away and the system is isolated.



Begin by placing q_1 at the origin. Since no other charges present this requires zero work. Next, bring q_2 to $(0,a)$, this process requires work. Note

$$\Delta U_1 = U_{12,f} - U_{12,i} = \frac{kq_1q_2}{a} - \cancel{\frac{kq_1q_2}{\infty}}^{\,0} = \frac{kq_1q_2}{a} = W_1$$

Next bring q_3 to its pictured position. The change in potential energy is again dependent only on the final configuration since q_3 is initially at $r=\infty$.

$$\Delta U_2 = U_f - U_i = \frac{kq_1q_3}{a} + \frac{kq_2q_3}{a} = W_2$$

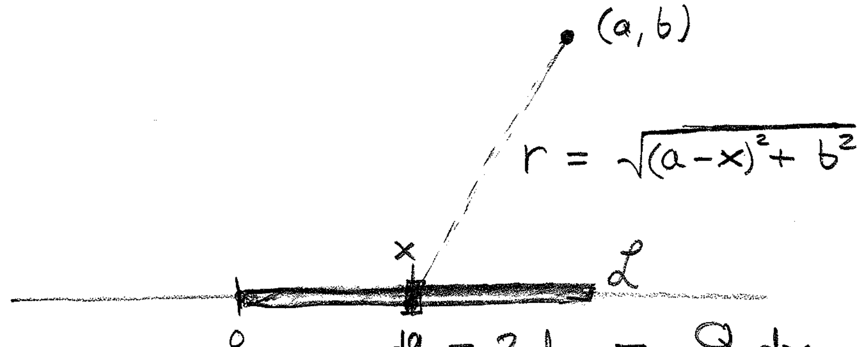
Notice ΔU_2 does not involve the (12) interaction since we already accounted for that energy in W_1. In total,

$$W = W_1 + W_2 = \underline{\frac{kq_1q_2}{a} + \frac{kq_1q_3}{a} + \frac{kq_2q_3}{a}}$$

this much work must be done <u>on</u> the $q_1q_2q_3$ system to bring q_1, q_2, q_3 from ∞ to the pictured final configuration.

(the $q_1q_2q_3$ system does $-W$ work in this process)

E33 Find the potential due to a line segment of charge where Q is uniformly distributed from $0 \le x \le L$.



(assume $a, b > 0$)

(for $b \neq 0$)

$$dq = \lambda dx = \frac{Q}{L} dx$$

$$V(a,b) = \int_{\mathcal{L}} \frac{k\,dq}{r} = \int_0^L \frac{k\lambda\,dx}{\sqrt{(a-x)^2+b^2}}$$

sidebar calculation

$$= \left(-k\lambda \ln\left|\frac{r}{b} + \frac{a-x}{b}\right|\right)\Big|_0^L$$

$$= \left[-k\lambda \ln\left(\tfrac{1}{b}(r+a-x)\right)\right]\Big|_0^L$$

$$= -k\lambda\left[\ln(r+a-x) - \ln(b)\right]\Big|_0^L$$

$$= -k\lambda \ln\left(\sqrt{(a-x)^2+b^2} + a - x\right)\Big|_0^L$$

$$= -k\lambda\left[\ln\left(\sqrt{(a-L)^2+b^2} + a - L\right) - \ln\left(\sqrt{a^2+b^2}+a\right)\right]$$

$$= \frac{kQ}{L} \ln\left[\frac{a + \sqrt{a^2+b^2}}{a - L + \sqrt{(a-L)^2+b^2}}\right]$$

Let $a - x = b\tan\theta$

$(a-x)^2 = b^2\tan^2\theta$

then $(a-x)^2 + b^2 =$

$= b^2(\tan^2\theta + 1)$

$= b^2\sec^2\theta$

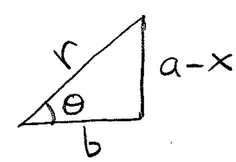
$dx = -b\sec^2\theta\, d\theta$

$$\int \frac{dx}{r} = \int \frac{-b\sec^2\theta\, d\theta}{\sqrt{b^2\sec^2\theta}}$$

$$= -\int \sec\theta\, d\theta$$

$$= -\int \frac{du}{u}$$

$$= -\ln|\underbrace{\sec\theta + \tan\theta}_{u}| + c$$



Replacing (a,b) with (x,y) we find

$$V(x,y) = \frac{kQ}{L} \ln\left[\frac{x + \sqrt{x^2+y^2}}{x - L + \sqrt{(x-L)^2+y^2}}\right]$$

As a check, think about $x = 0$ and $y \to \infty$ we have

$$V(0,y) \approx \frac{kQ}{L} \ln\left[\frac{y}{\sqrt{y^2+L^2} - L}\right] = \frac{kQ}{L}\left\{\ln(y) - \ln\left[\sqrt{y^2+L^2} - L\right]\right\}$$

$$\frac{dV}{dy} = \frac{kQ}{L}\left[\frac{1}{y} - \left(\frac{1}{\sqrt{y^2+L^2} - L}\right)\left[\frac{y}{\sqrt{y^2+L^2}}\right]\right].$$

E33 Continued

Trying to check answer in special case, $x=0, y\to\infty$,

$$\frac{dV}{dy} = \frac{kQ}{L}\left[\frac{1}{y} - \frac{y}{y^2+L^2-L\sqrt{y^2+L^2}}\right]$$

$$= \frac{kQ}{L}\left[\frac{y^2+L^2-L\sqrt{y^2+L^2}-y^2}{y(y^2+L^2-L\sqrt{y^2+L^2})}\right]$$

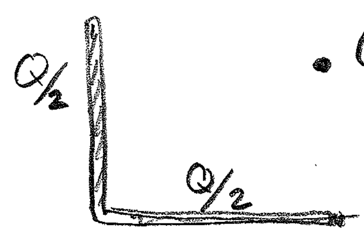
$$= \frac{kQ}{L}\left[\frac{L^2-L\sqrt{y^2+L^2}}{y^3+yL^2-Ly\sqrt{y^2+L^2}}\right]$$

$$\sim \frac{kQ}{L}\left[\frac{-L\sqrt{y^2+L^2}}{y^3}\right]$$ dominant terms as $y\to\infty$

$$\to \frac{-kQ}{y^2}$$

Thus we obtain $E_y = -\frac{\partial V}{\partial y} = \frac{kQ}{y^2}$ for $y \gg 0$.

In principle calculation of $\frac{\partial V}{\partial a}$ and $\frac{\partial V}{\partial b}$ should reproduce half of the $\vec{E}$ field from the problem you've solved for hwk. By symmetry I can write potential for L of charge

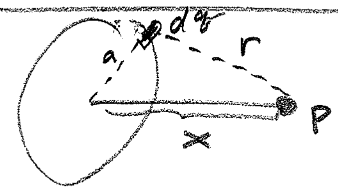


$$\boxed{V(a,b) = \frac{kQ}{2L}\ln\left[\frac{a+\sqrt{a^2+b^2}}{a-L+\sqrt{(a-L)^2+b^2}}\right] + \frac{kQ}{2L}\ln\left[\frac{b+\sqrt{a^2+b^2}}{b-L+\sqrt{a^2+(b-L)^2}}\right]}$$

If I haven't made some typo somewhere then $\vec{E}_{hwk} = \left\langle -\frac{\partial V}{\partial a}, -\frac{\partial V}{\partial b}\right\rangle$ where V is given above.

<u>Remark</u>: sometimes calculating V from $\int \frac{k\,dq}{r}$ and then finding $\vec{E} = -\nabla V$ is easier than direct computation of $\vec{E}$ from $\vec{E} = \int \frac{k\,dq}{r^2}\hat{r}$. I'd guess $\boxed{E33}$ is about the same trouble either way.

$\boxed{E34}$ (this is example 20.5 of Serway)
Find V and $\vec{E}$ due to ring of charge Q uniformly distributed in circle with radius a. Find V and $\vec{E}$ at distance x from center of ring along its axis



$$V = \int_{\text{ring}} \frac{k\,dq}{r}$$

$$= \int_0^{2\pi} \frac{k\lambda a\,d\theta}{\sqrt{x^2+a^2}}$$

$$= \frac{k\lambda a\,2\pi}{\sqrt{x^2+a^2}}$$

$$= \frac{kQ}{\sqrt{x^2+a^2}} \qquad \text{note } \lambda 2\pi a = Q.$$

$r = \sqrt{x^2+a^2}$

$\lambda = \frac{Q}{2\pi a} = \frac{dq}{ds}$

$ds = a\,d\theta$

$dq = \lambda ds = \lambda a\,d\theta$

Thus $\boxed{V(x) = \frac{kQ}{\sqrt{x^2+a^2}}}$

Calculate the $\vec{E}$-field at P by

$$E_x = -\frac{\partial V}{\partial x} = -kQ\frac{\partial}{\partial x}[x^2+a^2]^{-1/2}$$

$$= \frac{kQ}{2}(x^2+a^2)^{-3/2} \cdot 2x$$

$$\boxed{\vec{E} = \frac{kQx}{(\sqrt{x^2+a^2}\,)^3}\,\hat{x}}$$

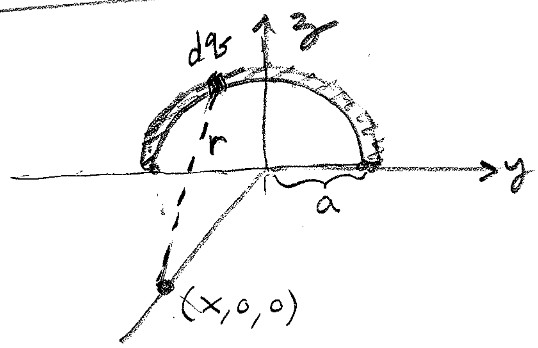
(By symmetry $E_y = E_z = 0$.)

Again $V \to \frac{kQ}{r}$ and $E \to \frac{kQ}{r^2}$ for $x >> a$.

Remark: $\boxed{E34}$ is more ballpark for tests than say $\boxed{E33}$.

QUESTION: in $\boxed{E34}$ I found $V = \dfrac{kQ}{\sqrt{x^2+a^2}}$.
Generally $\vec{E} = -\nabla V = -\langle \partial_x V, \partial_y V, \partial_z V \rangle$.
Why didn't I argue that $\dfrac{\partial V}{\partial y} = 0$ and $\dfrac{\partial V}{\partial z} = 0$ since V is only a function of x?

$\boxed{E35}$ Consider a half-circle of charge Q spread uniformly on $x=0$, $y^2+z^2=a^2$ for $z \geq 0$. Find the potential at $(x,0,0)$.



$$r = \sqrt{x^2+a^2}$$

$$V = \int \frac{k\,dq}{r} = \frac{k}{r}\int dq = \frac{kQ}{r}$$

$$\Rightarrow \boxed{V = \frac{kQ}{\sqrt{x^2+a^2}}}$$

We may again calculate $E_x = -\dfrac{\partial V}{\partial x} = \dfrac{kxQ}{(\sqrt{x^2+a^2})^3}$

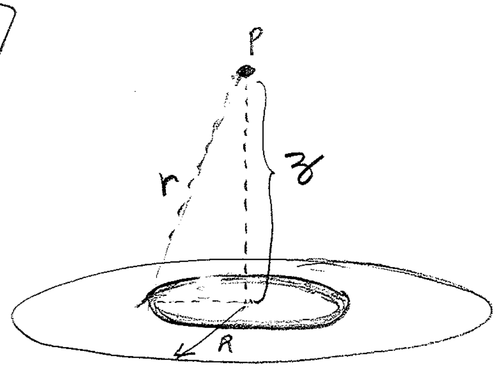
However, I cannot conclude that

$$\frac{\partial V}{\partial z} = 0.$$

(By symmetry it's clear that $E_y = 0$.)

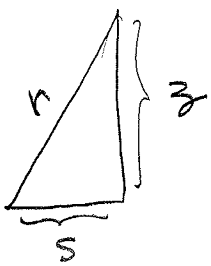
Why is it that $V = \dfrac{kQ}{\sqrt{x^2+a^2}}$ and yet $\dfrac{\partial V}{\partial z} \neq 0$? Note, it is physically clear that $E_z < 0$ if $Q > 0$, so we need $\dfrac{\partial V}{\partial z} \neq 0$.

E36



Find potential and electric field at P due to uniformly dist. charge Q over disk of radius R

Divide disk into little rings of radius s and thickness ds for $0 \le s \le R$. For each ring the charge dQ is equidistant from P



$$r = \sqrt{s^2 + z^2}$$

$$dQ = \sigma \, dA = \frac{Q}{\pi R^2} \cdot (2\pi s)\, ds$$

$$\Rightarrow dV = \frac{k\,dQ}{r} = \frac{2kQ}{R^2} \frac{s\,ds}{\sqrt{s^2+z^2}}$$

The net-potential at P is thus,

$$V = \int dV = \frac{2kQ}{R^2} \int_0^R \frac{s\,ds}{\sqrt{s^2+z^2}} \quad : \quad \begin{array}{l} u = s^2 + z^2 \\ u(0) = z^2 \\ u(R) = R^2 + z^2 \\ \tfrac{1}{2} du = s\,ds \end{array}$$

$$= \frac{kQ}{R^2} \int_{z^2}^{z^2+R^2} \frac{du}{\sqrt{u}}$$

$$= \boxed{\frac{2kQ}{R^2}\left[\sqrt{z^2+R^2} - z\right] = V(0,0,z)}$$

$$\left(\begin{array}{l} \text{for } z > 0 \\ \sqrt{z^2} = z \end{array}\right)$$

$$E_z = \frac{-dV}{dz} = \frac{-2kQz}{R^2\left(\sqrt{z^2+R^2}\right)^3} + \frac{2kQ}{R^2}.$$

If we suppose σ = constant and $R \to \infty$ then this requires $\frac{Q}{\pi R^2} \to \sigma$ as $R \to \infty$. With this in mind, $E_z = 2k\pi\sigma\left(1 - z(z^2+R^2)^{-3/2}\right)$

But, for any finite z the $-3/2$ term drops to zero as $R \to \infty$ hence

$$E_z \longrightarrow 2k\pi\sigma = \frac{2\pi\sigma}{4\pi\epsilon_0} = \frac{\sigma}{2\epsilon_0}$$

(which we found before via Gauss' Law!)

- See E23 p. 31 these notes. -

E37 (this is Ex. 20.6 in Serway)
An insulating solid sphere of radius R has total charge Q uniformly distributed. Find the potential everywhere

We calculated from Gauss' Law that

$$E(r) = \begin{cases} \dfrac{kQ}{R^3} r & : \ 0 \leq r \leq R \\ \dfrac{kQ}{r^2} & : \ r \geq R \end{cases}$$

and $\vec{E} = E(r)\hat{r}$, the field points isotropically outward from the center of the charged sphere. (See Ex 19.10 in Serway, pg. 627-628, we derived this in lecture as well). I can calculate the potential by the formula $\vec{E} = -\nabla V = -\dfrac{dV}{dr}\hat{r}$ in this context. We find from integration that

$$V(r) = \begin{cases} \dfrac{-1}{2}\dfrac{kQ}{R^3} r^2 + C_1 & : \ 0 \leq r \leq R \\ \dfrac{kQ}{r} + C_2 & : \ r \geq R \end{cases}$$

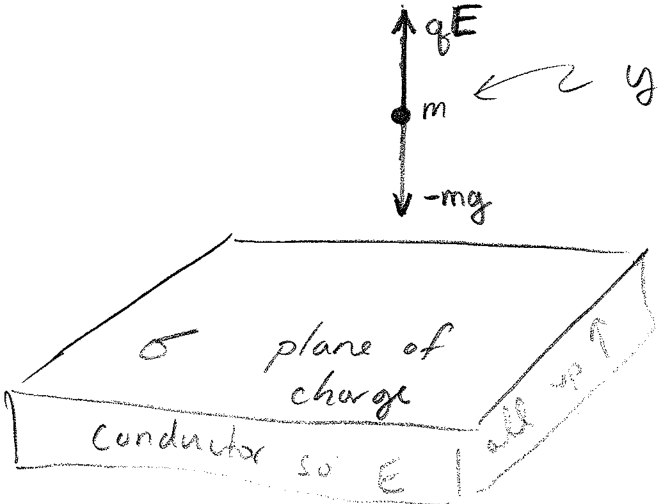
If we assume $V(\infty) = 0$ then $\Rightarrow C_2 = 0$.
Then $V(R) = \dfrac{kQ}{R} = \dfrac{-1}{2}\dfrac{kQ}{R^3}R^2 + C_1$ follows from assuming V is a continuous fnct. of r.
Thus, $C_1 = \dfrac{kQ}{R} + \dfrac{1}{2}\dfrac{kQ}{R} = \dfrac{3kQ}{2R}$ and we have

$$\boxed{V(r) = \begin{cases} \dfrac{3kQ}{2R} - \dfrac{kQ}{2R^3} r & : \ 0 \leq r \leq R \\ \dfrac{kQ}{r} & : \ r \geq R \end{cases}}$$

If we made $V = 0$ at some other r-value then that would change C_2 and consequently C_1.

Remark: discontinuity in V signals a concentration of localized charge. Think about $V = \dfrac{kQ}{r}$.

E 38 Suppose $y = y_0 + 3t^2$ for a particle with mass $m = 1\,\text{kg}$ and charge $q = \left(\frac{1}{9\times10^9}\right)C$. Suppose y is the vertical direction and we do consider gravity. If the only other force acting on m is the electric force due to the pictured plane charge (conductor) below then what is σ? (assume only vertical motion)



$\longleftarrow y = y_0 + 3t^2$

$$ma = qE - mg$$

meaning:

$$m\frac{d^2y}{dt^2} = \frac{q\sigma}{\epsilon_0} - mg$$

We calculate, $\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}(3t^2)\right) = \frac{d}{dt}(6t) = 6 \Rightarrow \underbrace{\ddot{y} = 6\frac{m}{s^2}}_{\text{(putting back units)}}$

Hence, solving for σ,

$$\sigma = \frac{\epsilon_0}{q}\left[\left(6\frac{m}{s^2} + 9.8\frac{m}{s^2}\right)\cdot 1\,\text{kg}\right]$$

$$= \left(\frac{8.85\times10^{-12}\frac{C^2}{N\cdot m^2}}{\frac{1}{9\times10^9}C}\right)\left(15.8\,\cancel{\frac{m\,kg}{s^2}}\right) \qquad N = \frac{kg\,m}{s^2}$$

$$\cong 1.26\,\frac{C}{m^2}$$

Thus $\boxed{\sigma = \frac{dQ}{dA} = 1.26\,\frac{C}{m^2}}$

__Remark:__ I've not written notes about $\vec{E} = 0$ inside conductor and equipotentials. I will discuss this in lecture qualitatively. Read 655-656 of Serway.