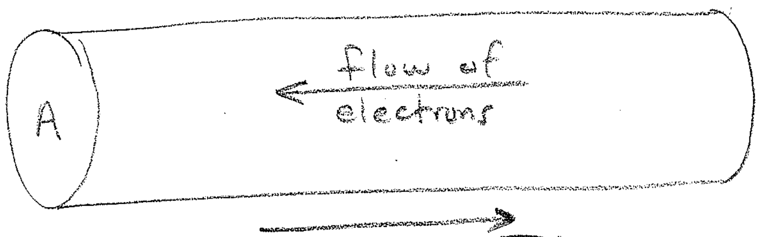
RESISTIVITY & RESISTANCE ...

I'm omitting detailed discussion of the resistivity in these notes for Fall 2010. I'll merely state the general formulae and summarize a few interesting facts / figures.



$\vec{E} = \rho \vec{J}$ ← ρ: resistivity

$I = JA$ (I: current, J: current density, A: cross-sectional area of wire)

Units $\frac{C}{s}$ = Amp = A (for Ampere)

$$I = \underbrace{\frac{dQ}{dt}}_{} = qnAV_d$$

V_d = drift speed of free charge = $\underbrace{3.5}_{\text{copper}}$ mm/s

current is charge per unit time cutting through some cross-section. Usually we consider wire so details of don't so much matter.

$$I = \int_S \vec{J} \cdot d\vec{A}$$

When $\vec{E}$-field applied to conductor it generates surface charge which moves the free electrons on the avg. in the direction opposite the field. Because the electrons bump into the lattice of the conductor energy is lost and potential drops ; $V = E\Delta L$

If V/I = constant then we call this constant the <u>resistance</u> ; $\boxed{V = IR}$ OHM's LAW.

The <u>resistivity</u> is a characteristic of a conductor which allows us to quantify R from geometric data ; $R = \rho \frac{L}{A}$ (for wire)

← OHM defined this by analogy to heat conduction...

The resistivity ρ is a function of temperature. It increases linearly with T for metals. This is why light-bulbs flash bright at start then dim, to begin

$$R_{\substack{\text{light bulb} \\ \text{first on}}} < R_{\substack{\text{light bulb} \\ \text{heated up}}} \quad \text{since} \quad \underset{\text{fixed}}{V} = \underset{\text{varying}}{IR}$$

$\xrightarrow{40\,ms}$

Incidentally, this is opposite the classical statistical mechanics for a conductor predicts. One observed instance of quantum mechanics is this behaviour of ρ with T. It has an explaination in the theory of quantum mechanical statistical mechanics.

<u>FACT</u>: Bigger wires have less resistance / length and hence drop less voltage per length.

For example, 14 gauge wire has $0.0082\ \Omega/m$ ← an "Ohm"

So, if we put $I = 2A$ through

- 1-meter of 14 gauge wire there is a voltage drop of $V = IR = (2A)(8.2\,m\Omega) = \underline{16.2\ mV}$. ← milli $= 10^{-3}$
- 100-meters $\Rightarrow R_{wire} = 0.82\ \Omega$
  $\Rightarrow I = 10A$ gives $\underline{V_{lost} = 8.2V}$

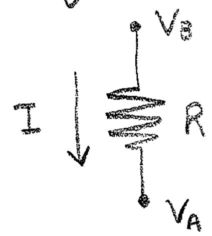
<u>Note</u>: $R = \rho \frac{L}{A}$ and

$A = 3.309\ mm^2$ ← 12 gauge
$A = 2.081\ mm^2$ ← 14 gauge
$A = 0.3255\ mm^2$ ← 22 gauge

for fixed length, 12 gauge has

Power

Suppose a resistor R has current I flowing through it and a voltage $V = V_B - V_A$.



OHM's Law says

$$V = IR$$

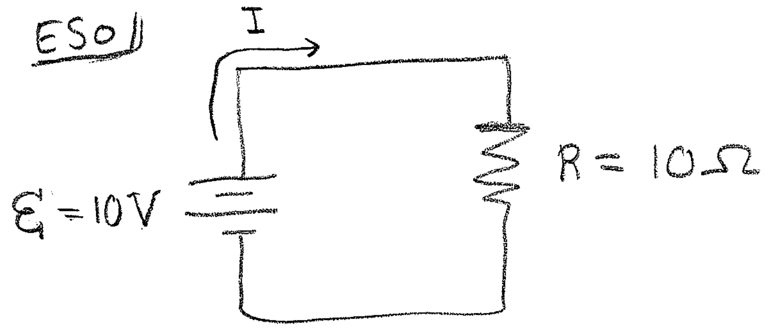
Think about it, dQ flows from potential V_B to V_A and it follows $dU = (V_B - V_A)\,dQ$ divide by the time of the motion, dt and obtain

$$\frac{dU}{dt} = (V_B - V_A)\frac{dQ}{dt} = VI$$

Usually we say $V_B - V_A = V$ so we find the simple formula for $P = \frac{dU}{dt} = \underline{\text{power}}$

$$\underbrace{\boxed{P = IV = I^2R = V^2/R}}_{\text{rate of energy loss from charge flowing through the resistor.}}$$

ESo)



$$V = IR$$

$$I = \frac{10V}{10\Omega} = 1A.$$

$$P = IV = (1A)(10V)$$

$$\Rightarrow \boxed{P = 10\,W}$$

$$\underbrace{W = \text{Watt}}_{\text{Units of power.}} = A \cdot V = \frac{C}{s} \cdot \frac{J}{C} = \frac{J}{s}$$

energy/time

DC CIRCUIT ANALYSIS

THEORY: ① SUM OF VOLTAGE DROPS AROUND CLOSED LOOP TOTALS TO ZERO (Kirchoff's Voltage Rule)

② SUM OF CURRENTS LEAVING A NODE IS EQUAL TO ZERO (CONSERVATION OF CHARGE)

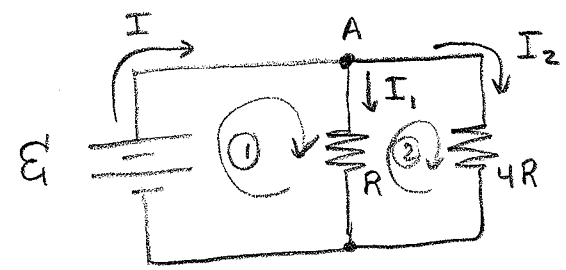
③ Ohm's Law Applies

④ Voltage Sources have <u>fixed</u> voltage.

⑤ Current Sources have <u>fixed</u> current.

We'll only have to deal with ④ in most cases but ⑤ is also interesting to engineers.

ES1 Suppose $\mathcal{E}_1 = 10\,V$, $R = 1k\Omega$ find I.



(I) Loop 1 $\quad \mathcal{E}_1 - I_1 R = 0$

(II) Loop 2 $\quad I_1 R - I_2(4R) = 0$

(III) node A $\quad I_1 + I_2 - I = 0$

more than needed to solve.

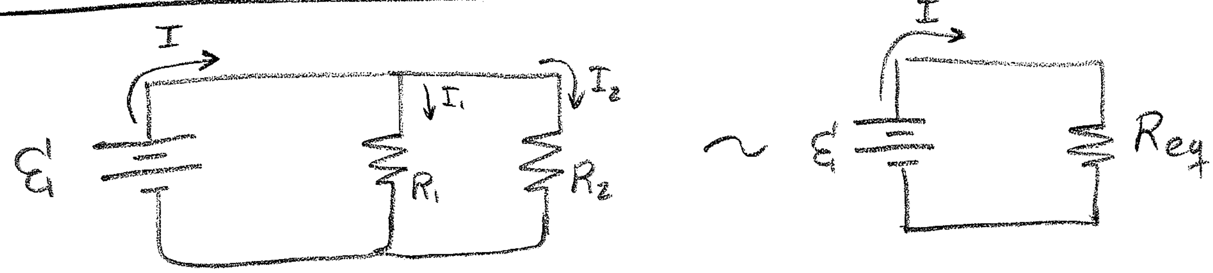
(IV) BIG LOOP $\quad \mathcal{E}_1 - I_2 4R = 0$

Use (I) to see $I_1 = \mathcal{E}_1/R = 10V/1k\Omega = 10\,mA$.

Use (IV) to see $I_2 = \mathcal{E}_1/4R = 10V/4k\Omega = 2.5\,mA$.

Therefore, by (III) we find $I = I_1 + I_2 = \boxed{12.5\,mA}$

Equivalent Circuit Shortcut



$$\mathcal{E} = I_1 R_1 = I_2 R_2$$

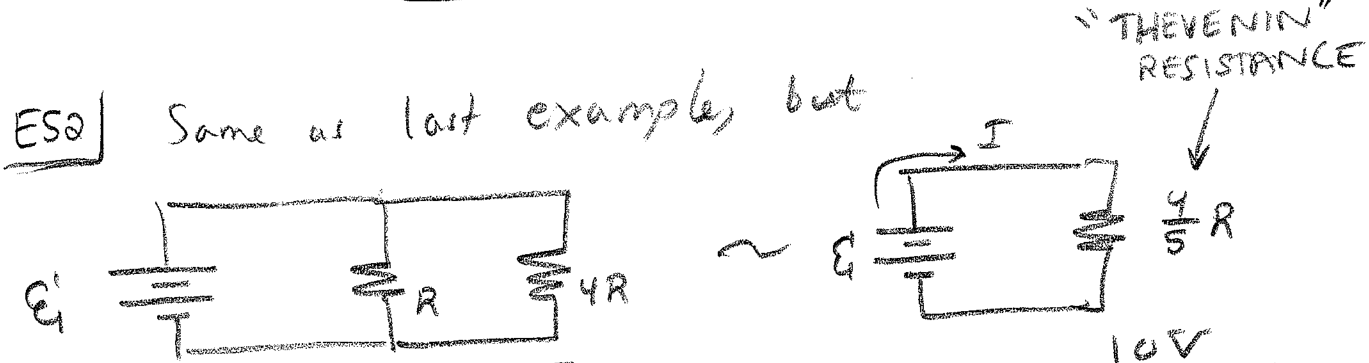
$$\& \quad I = I_1 + I_2$$

But, $\mathcal{E} = I R_{eq} = (I_1 + I_2) R_{eq}$

$$= \left(\mathcal{E}/R_1 + \mathcal{E}/R_2 \right) R_{eq}$$

$$\Rightarrow \boxed{R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 R_2}{R_1 + R_2}}$$

ES2 Same as last example, but



$$R_{eq} = \frac{4R^2}{5R}$$

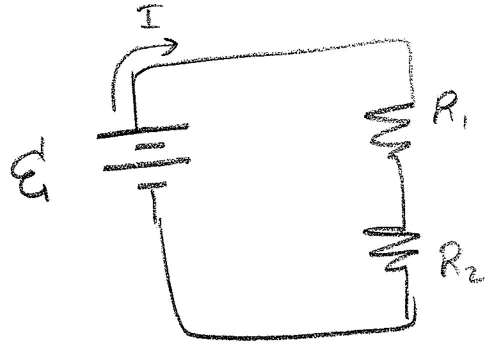
$$\therefore \; I = \frac{10\,V}{\frac{4}{5}\,1k\Omega}$$

$$I = \frac{50}{4}\,mA$$

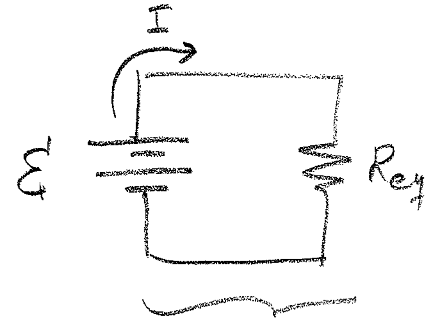
$$\therefore \; \boxed{I = 12.5\,mA}$$

Remark: the technique of equivalent resistance is nice where applicable BUT the loops / current conservation method is more general and will solve harder problem with multiple loops.

SERIES EQUIVALENT



$\sim$



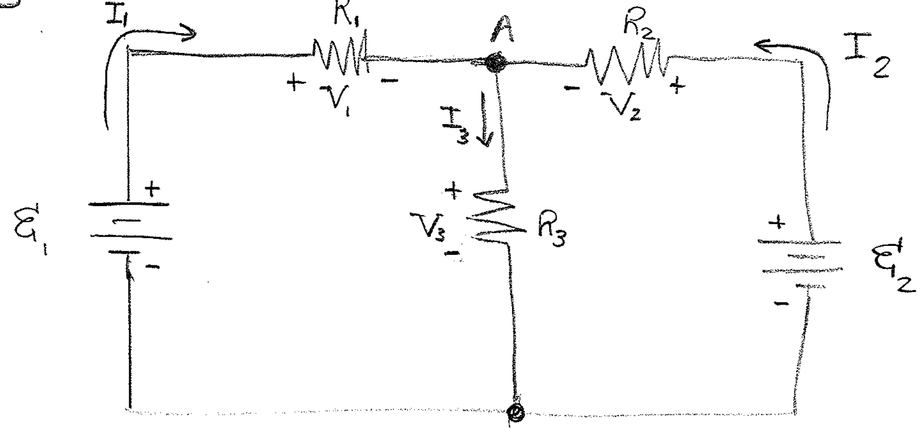
$$\mathcal{E}_1 - I R_{eq} = 0$$
$$I = \mathcal{E}_1 / R_{eq}$$

$$\mathcal{E}_1 - I R_1 - I R_2 = 0$$
$$\hookrightarrow \quad I = \frac{\mathcal{E}_1}{R_1 + R_2}$$

$$\therefore \quad I = \frac{\mathcal{E}_1}{R_1 + R_2} = \frac{\mathcal{E}_1}{R_{eq}} \Rightarrow \boxed{R_{eq} = R_1 + R_2}$$

73

E53 Find I_1, I_2, I_3 for the given $\mathcal{E}_1, \mathcal{E}_2, R_1, R_2, R_3$.



(I) RIGHT LOOP $\quad I_3R_3 + I_2R_2 - \mathcal{E}_2 = 0$

(II) LEFT LOOP $\quad \mathcal{E}_1 - I_1R_1 - I_3R_3 = 0$

(III) CURRENT CONS. AT NODE A $\quad I_1 + I_2 - I_3 = 0$

(IV) BIG LOOP $\quad \mathcal{E}_1 - I_1R_1 + I_2R_2 - \mathcal{E}_2 = 0$

As a matrix eq$^{\underline{n}}$ we have:

$$\begin{bmatrix} 0 & R_2 & R_3 \\ -R_1 & 0 & -R_3 \\ 1 & 1 & -1 \\ -R_1 & R_2 & 0 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} \mathcal{E}_2 \\ -\mathcal{E}_1 \\ 0 \\ \mathcal{E}_2 - \mathcal{E}_1 \end{bmatrix}$$

Notice the redundancy of the Big Loop Eq$^{\underline{n}}$ is manifest with the matrix notation (row 4 = row 1 + row 2)

I'll focus on eq$^{\underline{n}}$'s (I), (II) & (III),

$$\left[\begin{array}{ccc|c} 0 & R_2 & R_3 & \mathcal{E}_2 \\ -R_1 & 0 & -R_3 & -\mathcal{E}_1 \\ R_1 & R_1 & -R_1 & 0 \end{array}\right]$$

$\leftarrow$ can solve by many methods. I'll illustrate Kramer's Rule,

$$I_1 = \frac{\det\begin{bmatrix} \mathcal{E}_2 & R_2 & R_3 \\ -\mathcal{E}_1 & 0 & -R_3 \\ 0 & R_1 & -R_1 \end{bmatrix}}{\det\begin{bmatrix} 0 & R_2 & R_3 \\ -R_1 & 0 & -R_3 \\ R_1 & R_1 & -R_1 \end{bmatrix}} = \frac{\mathcal{E}_2 R_1 R_3 - R_2 R_1 \mathcal{E}_1 - R_3 R_1 \mathcal{E}_1}{-R_2 R_1 R_1 - R_2 R_1 R_3 - R_3 R_1 R_1}$$

Continuing the algebra from last problem,

$$\boxed{I_1 = \frac{\mathcal{E}_2 R_3 - \mathcal{E}_1 R_2 - \mathcal{E}_1 R_3}{-R_1 R_2 - R_2 R_3 - R_1 R_3}}$$

Now that we have one current the others will be easier to derive, by II,

$$I_3 = \frac{1}{R_3}\left(\mathcal{E}_1 - I_1 R_1\right)$$

$$\Rightarrow \boxed{I_3 = \frac{1}{R_3}\left[\mathcal{E}_1 + R_1\left(\frac{\mathcal{E}_2 R_3 - \mathcal{E}_1 R_2 - \mathcal{E}_1 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3}\right)\right]}$$

Finally, for I_2 we use III,

$$I_2 = I_3 - I_1$$

$$\Rightarrow \boxed{I_2 = \frac{1}{R_3}\left[\mathcal{E}_1 + R_1\left(\frac{\mathcal{E}_2 R_3 - \mathcal{E}_1 R_2 - \mathcal{E}_1 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3}\right)\right] + \frac{\mathcal{E}_2 R_3 - \mathcal{E}_1 R_2 - \mathcal{E}_1 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3}}$$

Now, of course I give problems with numbers usually so they seem easier than this. For example, suppose we're given

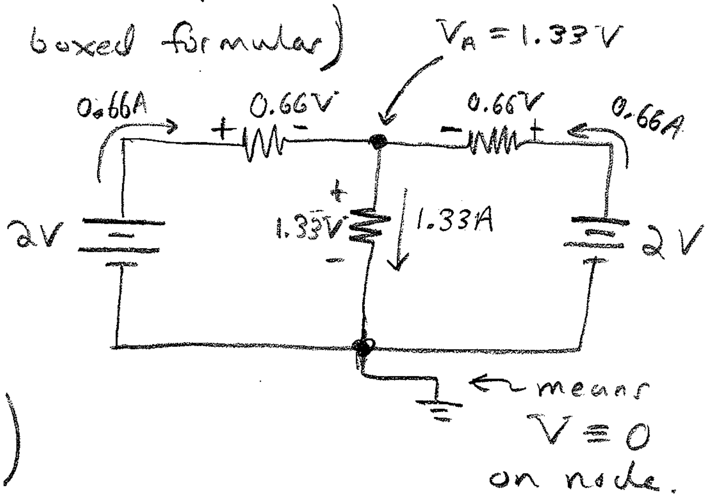
$$\mathcal{E}_1 = 2V, \quad \mathcal{E}_2 = 2V \quad \text{and} \quad R_1 = R_2 = R_3 = 1\,\Omega$$

we find that (plus boxed formulas)

$$I_1 = \frac{1}{3}\frac{\mathcal{E}_1}{R} = \frac{2}{3}\,A.$$

$$I_2 = \frac{1}{3}\frac{\mathcal{E}_1}{R} = \frac{2}{3}\,A$$

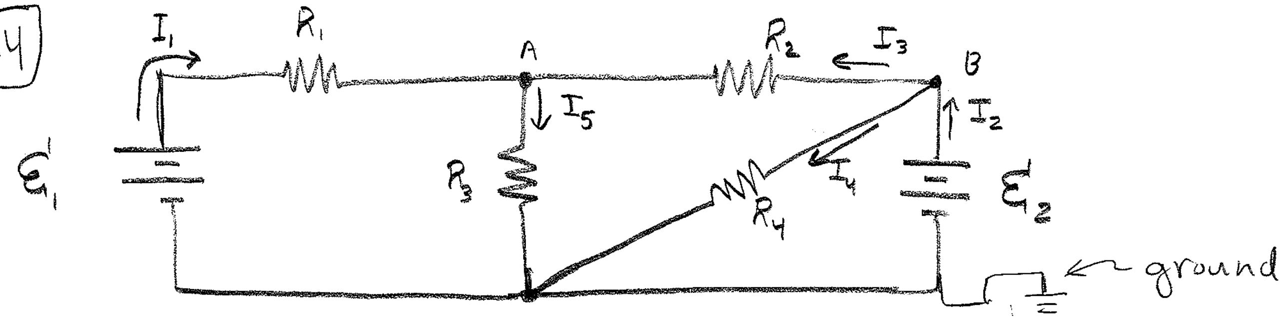
$$I_3 = \frac{2}{3}\frac{\mathcal{E}_1}{R} = \frac{4}{3}\,A$$



(So, my general answer checks against at least this case)

I'll set-up, but not solve, another loopy circuit

E54



Loop LEFT ↻	$\mathcal{E}_1 - I_1 R_1 - I_5 R_3 = 0$	5 eqⁿ's in 5 unknown currents.
UPPER Triangular ↻	$-I_3 R_2 - I_5 R_3 + I_4 R_4 = 0$	
LOWER TRIANGLE ↻	$\mathcal{E}_2 - I_4 R_4 = 0$	
currents at A	$I_1 + I_3 - I_5 = 0$	
currents at B	$I_2 - I_3 - I_4 = 0$	

Usually $\mathcal{E}_1, \mathcal{E}_2, R_1, R_2, R_3, R_4$ would be given then we could derive the currents from Kirchoff's Voltage Law and/or current conservation.

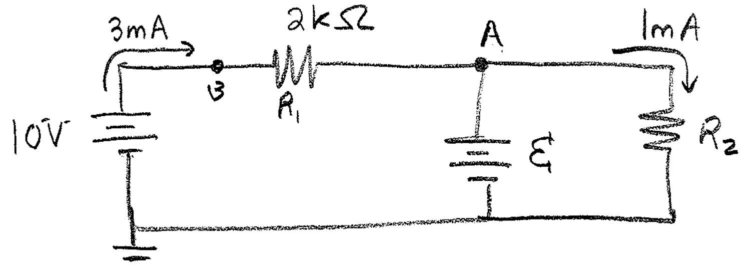
Remark: can also use V_A and V_B as variables. Once a ground is defined then absolute voltages are of interest. For this circuit clearly $V_B = \mathcal{E}_2$ hence $I_4 = \mathcal{E}_2/R_4$. However, I_2 & I_3 depend on the choices made for R_2, R_3, R_1 and $\mathcal{E}_1$

Remark: the ~~equivalent~~ resistance idea not so helpful here. (which isn't to say $\nexists$ a more sophisticated idea of Thevenin resistance, just saying I'm not advocating it for circuits with multiple voltage sources)

Easier, algebraically lucid, examples

ESS

Find V_A given the circuit pictured below,



Also what must $\mathcal{E}'$ and R_2 be given this data?

Sol$^{\underline{n}}$: Notice the voltage at A is V_A and by Ohm's Law we lose $I_1R_1 = (3mA)(2k\Omega) = 6\,V$ across the $2k\Omega$ resistor hence $\underline{V_A = 10V - 6V = 4V}$.

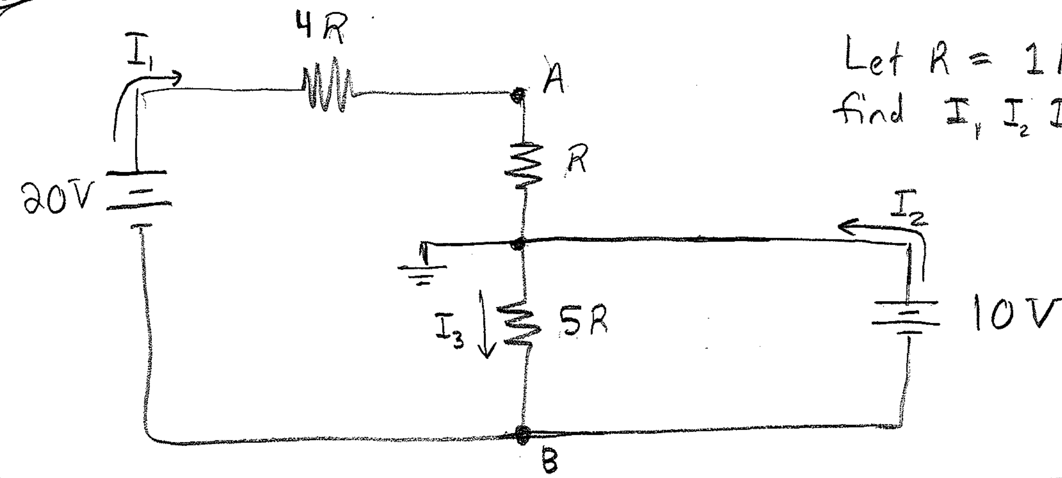
thought process in detail

I used that $V_B = 10V$ since the voltage source gives $10\,V$ difference between ground and its upper ~~terminal~~; then technically Ohms Law is $\Delta V = I_1R_1 = V_B - V_A$ thus $V_A = V_B - I_1R_1 = 10V - 6V = 4V$

Now, $V_A = \mathcal{E}'$ by inspection thus $\boxed{\mathcal{E}' = 4V}$

Finally $(1mA)R_2 = 4V$ so we deduce $\boxed{R_2 = 4k\Omega}$

Remark: same physics as before, just no linear algebra / 3 eq$^{\underline{n}}$'s & 3 unknowns.



Let $R = 1M\Omega$
find I_1 I_2 I_3 and V_A, V_B

$\underline{Sol^n}$: Again the $\frac{\perp}{}$ means $V \equiv 0$ at that point. Ohm's Law says that $V_A - 0 = I_1 R$. Also we expect $0 - V_B = I_3(5R)$ from Ohm's Law on 5R. We also note $I_1 + I_2 = I_3$. Now, we could use loop-analysis solve the circuit, BUT we don't need to. Instead we'll work our way from the 10V source over,

<u>OBSERVATION</u>: $10V = 0 - V_B$ $\therefore$ $\underline{V_B = -10V}$

$$\therefore \quad I_3 = 10V/5M\Omega = \underline{2\mu A = I_3}.$$

<u>OBSERVATION</u>: series combo of 4R & R $\Rightarrow R_{eq} = 5M\Omega$

thus $\underbrace{20V - 5RI_1 - 10V}$ $= 0$ ← (LEFT LOOP Kirchoff's Voltage Rule)

$$I_1 = \frac{10V}{5M\Omega} = \underline{2\mu A = I_1}$$

It follows that $I_2 = I_3 - I_1 = 2\mu A - 2\mu A = 0$.

To summarize,

$$\boxed{\begin{array}{l} V_A = 2V, \quad V_B = -10V \\ I_1 = 2\mu A, \quad I_2 = 0, \quad I_3 = 2\mu A \end{array}}$$

Remark: Sometimes physical intuition allows us to circumvent algebra, but, not always...