Energy Diagrams (useful for gravitation motion, chemistry, solid state physics, ... important concept!)

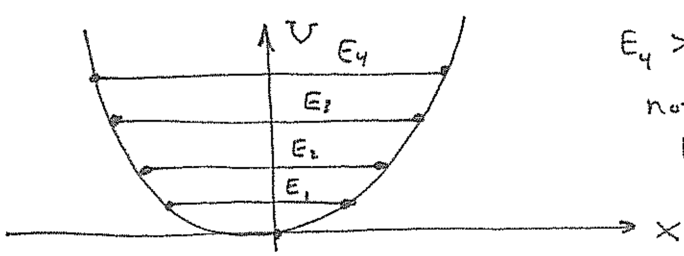
Given a graph of the potential energy function verses x for a one-dimensional system we can easily extract much data about the possible motions of the system.

I'll assume energy $E = K + U$ is conserved, but this discussion is easily twisted to the nonconservative case. The crucial observations are as follows:

$$F = -\frac{dU}{dx}$$ ← can see direction of force from slope of U vs. x graph

$$K = \frac{1}{2}mv^2 \geq 0$$ thus $E = E_o$ is not allowed to have $U < E_o$ since $E = K + U \geq U$

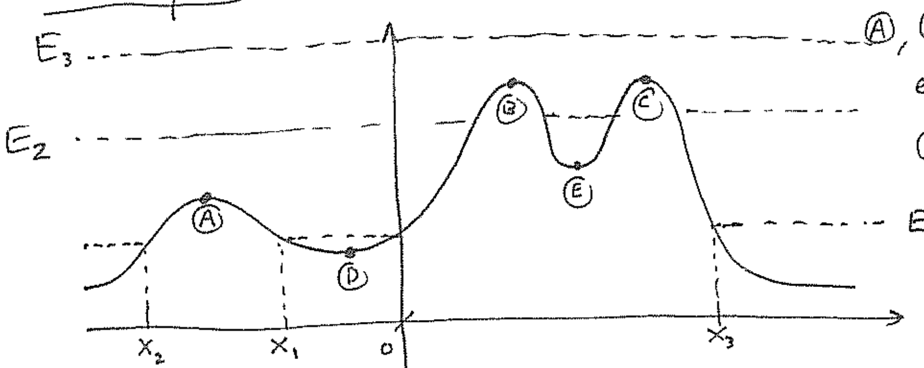
Example) $U = \frac{1}{2}kx^2$



$E_4 > E_3 > E_2 > E_1$

note, $K = 0$ where the lines intersect $U = \frac{1}{2}kx^2$.

Example) (Discuss)



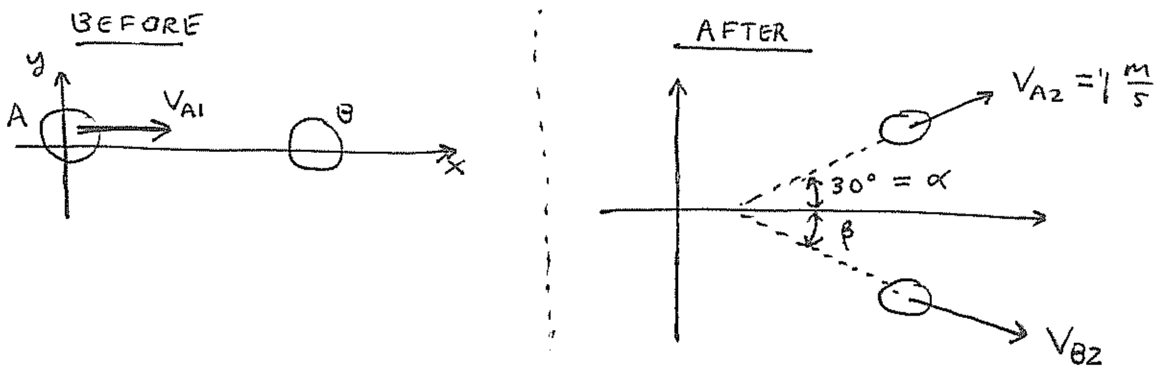
Ⓐ, Ⓑ, Ⓒ unstable equilibriums.

Ⓓ, Ⓔ stable equilibriums.

If we have total energy E_1 then either $x \leq x_2$, or $x_1 \leq x \leq 0$ or $x_3 \leq x$. However, $0 \leq x \leq x_3$ is not classically permitted.

Momentum & Collisions

Example: two students on spring break in Canada are sitting on an ice pond in lawn chairs. One of them has an umbrella and a strong wind sends student Ⓐ on a collision course with student Ⓑ. Supposing $m_A = 5\,kg$ and $\vec{V}_{A1} = (2\,m/s)\,\hat{i}$ (after the wind sets Ⓐ in motion) if $m_B = 3\,kg$ is initially at rest and after the collision $V_{A2} = 1\,m/s$ at $\alpha = 30°$ then what is $\vec{V}_{B2}$?



Sol^n: conserve momentum.

$$\vec{P}_1 = m_A \vec{V}_{A1} = \vec{P}_2 = m_A \vec{V}_{A2} + m_B \vec{V}_{B2}$$

Break it down into components.

$$\vec{V}_{A2} = (\cos 30°)\, V_{A2}\, \hat{i} + (\sin 30°)\, V_{A2}\, \hat{j}$$

$$\vec{V}_{B2} = (\cos\beta\, V_{B2})\, \hat{i} - (\sin\beta\, V_{B2})\, \hat{j}$$

This gives us,

$$\hat{i}: \quad m_A V_{A1} = m_A \cos 30°\, V_{A2} + m_B \cos\beta\, V_{B2}$$

$$\hat{j}: \quad 0 = m_A \sin 30°\, V_{A2} - m_B \sin\beta\, V_{B2}$$

Solve for $V_{B2x} = \cos\beta\, V_{B2} = \dfrac{m_A V_{A1} - m_A \cos 30°\, V_{A2}}{m_B} = 1.89\, m/s.$

Likewise $V_{B2y} = -\sin\beta\, V_{B2} = -\dfrac{m_A \sin 30°\, V_{A2}}{m_B} = -0.83\, m/s,$

Thus $\boxed{\vec{V}_{B2} = \langle 1.89\, m/s,\ -0.83\, m/s \rangle}$ this gives magnitude (a.k.a. speed) of $V_{B2} = 2.1\, m/s$ and $\beta = \tan^{-1}\left(\frac{-0.83}{1.89}\right)$

$\beta = -24°$

Center of Mass

Concept: we can replace a system of n-particles with masses $m_1, m_2, .., m_n$ subject to forces $\vec{F}_1, \vec{F}_2, .., \vec{F}_n$ by a particle with mass $M = m_1 + m_2 + \cdots + m_n$ positioned at the center of mass $\vec{r}_{cm} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n)$ where $\vec{r}_1, \vec{r}_2, .., \vec{r}_n$ denote the position vectors of $m_1, m_2, .., m_n$ respectively. In particular, if $\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n$ then

$$\underbrace{M\vec{a}_{cm} = \vec{F}}$$

this is true because $m_1\vec{a}_1 = \vec{F}_1$ etc... and

$$\begin{aligned} M\vec{a}_{cm} &= M\frac{1}{M}\left(m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n\right) \\ &= m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n \\ &= \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n . \end{aligned}$$

Therefore, if $\vec{P} = M\vec{V}_{cm} = M\frac{d\vec{r}_{cm}}{dt} = \vec{P}_1 + \vec{P}_2 + \cdots + \vec{P}_n$ where $\vec{P}_1 = m_1\vec{V}_1, \vec{P}_2 = m_2\vec{V}_2, \ldots, \vec{P}_n = m_n\vec{V}_n$ then we find

$$\frac{d\vec{P}}{dt} = \vec{F}$$

Note, if all the forces are internal to the system then $\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n = 0$ (By 3rd Law of Newton)

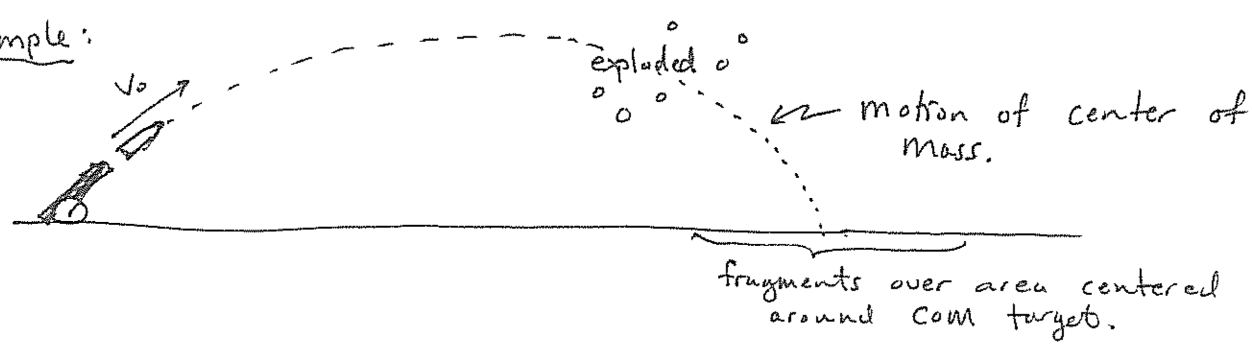
Hence, given $\vec{F}_{net} = 0$ (no external forces)

$$\frac{d\vec{P}}{dt} = 0 \iff \frac{d}{dt}\left(\vec{P}_1 + \vec{P}_2 + \cdots + \vec{P}_n\right) = 0$$

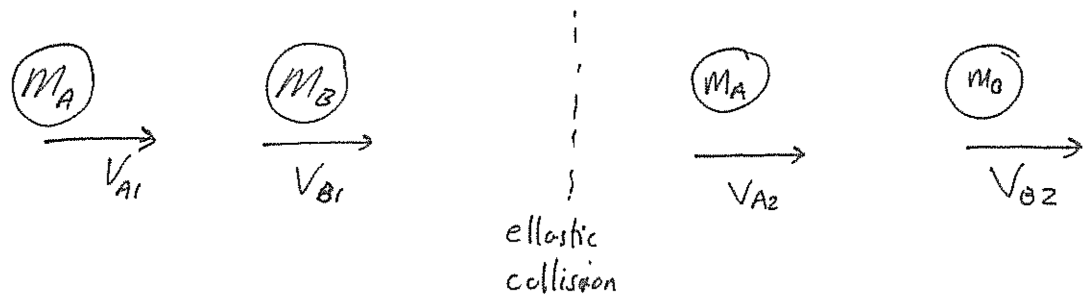
$\iff$ total momentum $\vec{P}$ is conserved.

Remark: C.O.M. is useful for understanding conservation of momentum and also for viewing systems macroscopically.

Example:



Ellastic Collisions



Generally we have <u>two</u> conservation eq^{ns},

$$m_A \vec{V}_{A1} + m_B \vec{V}_{B1} = m_A \vec{V}_{A2} + m_B \vec{V}_{B2}$$

$$\frac{1}{2} m_A V_{A1}^2 + \frac{1}{2} m_B V_{B1}^2 = \frac{1}{2} m_A V_{A2}^2 + \frac{1}{2} m_B V_{B2}^2$$

<u>Simple case to study</u>: Suppose B is at <u>rest</u> initially so $V_{B1} = 0$. Discuss the interesting features of the resulting motion (assume 1-dim'l motion)

<u>Algebra</u>!

$$m_A V = m_A V_A + m_B V_B$$

$$\frac{1}{2} m_A V^2 = \frac{1}{2} m_A V_A^2 + \frac{1}{2} m_B V_B^2$$

(let $V_{A1} = V$, $V_{A2} = V_A$, $V_{B2} = V_B$ and there are 1-dim'l <u>vectors</u> they can be negative to indicate leftward direction)

Can show that: (see pg. 6)

$$V_A = \left(\frac{m_A - m_B}{m_A + m_B}\right) V$$

$$V_B = \left(\frac{2 m_A}{m_A + m_B}\right) V$$

where it also can be shown $V_B = V + V_A$ hence

$$\underbrace{V = V_B - V_A}$$

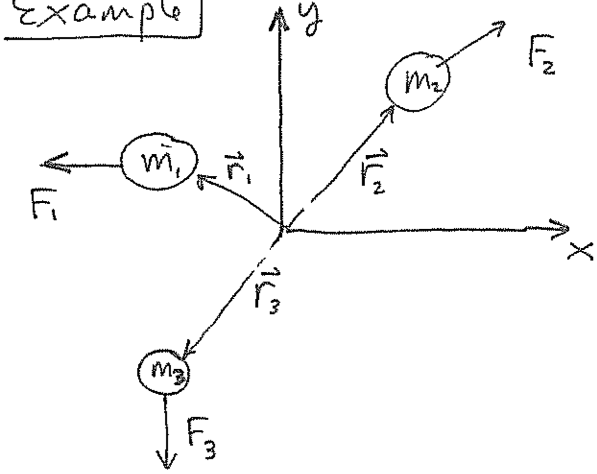
relative velocity opposite before & after collision

since $V_{B1} - V_{A1} = -V$ verse

$V_{B2} - V_{A2} = V_B - V_A = V.$

$$\boxed{\vec{V}_{\text{relative before}} = -\vec{V}_{\text{relative after}} \quad \text{for Ellastic Collisions}}$$

Example



$$\vec{r}_{cm} = \frac{1}{m_1+m_2+m_3}\left(m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3\right)$$

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

$$\vec{a}_{cm} = \frac{d^2\vec{r}_{cm}}{dt^2}$$

Suppose $\vec{F}_1 = -F_0\hat{i}$, $\vec{F}_2 = F_0(\hat{i}+\hat{j})$ and $\vec{F}_3 = -F_0\hat{j}$.

where $\vec{r}_1 = \langle -1, 1\rangle r_0$, $\vec{r}_2 = \langle 1, 2\rangle r_0$, $\vec{r}_3 = \langle -1, -2\rangle r_0$

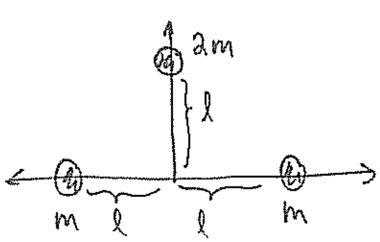
Let $m_1 = m_2 = m_3 = 1kg$ and $r_0 = 1m$ and $F_0 = 1N$.

Suppose m_1, m_2, m_3 are attached to a rigid frame which has very small mass. Find the motion. (assume rest at $t=0$.)

$$\begin{aligned}\vec{F} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\ &= -\hat{i} + \hat{i} + \hat{j} - \hat{j} \\ &= 0 \quad \therefore \quad \vec{a}_{cm} = 0 \Rightarrow \vec{r}_{cm} = \text{constant}\end{aligned}$$

$$\begin{aligned}\vec{r}_{cm} &= \frac{1}{3}\left(\vec{r}_1 + \vec{r}_2 + \vec{r}_3\right) \\ &= \frac{1}{3}\left(\langle -1, 1\rangle + \langle 1, 2\rangle + \langle -1, -2\rangle\right) m \\ &= \frac{1}{3}\langle -1, 1\rangle m\end{aligned}$$

Ex

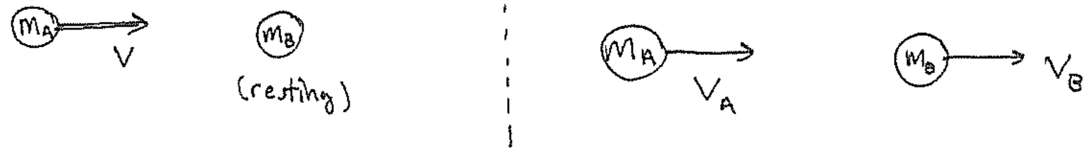


$$\begin{aligned}\vec{r}_{cm} &= \frac{1}{4m}\left(m\ell\hat{i} - m\ell\hat{i} + 2m\hat{j}\right) \\ &= \frac{2m\ell}{4m}\hat{j}\end{aligned}$$

$$\boxed{\vec{r}_{cm} = \frac{\ell}{2}\hat{j}}$$

ONE-DIMENSIONAL ELLASTIC COLLISION (PROOF)

(BEFORE) | (AFTER)



Supposing the collision was elastic yields:

I $\quad \frac{1}{2} m_A V^2 = \frac{1}{2} m_A V_A^2 + \frac{1}{2} m_B V_B^2$

The system is isolated, so $F_{net} = 0$ and momentum of m_A, m_B is conserved,

II $\quad m_A V = m_A V_A + m_B V_B$

Notice that I yields,

$$\underline{V^2 = V_A^2 + \frac{m_B}{m_A} V_B^2} \quad \text{III}$$

I can solve II for V_B to obtain $\underline{V_B = \frac{m_A (V - V_A)}{m_B}}$ IV

Substituting IV into III gives

$$V^2 = V_A^2 + \frac{m_B}{m_A}\left(\frac{m_A (V - V_A)}{m_B}\right)^2$$

$$\Rightarrow V^2 - V_A^2 = \frac{m_A}{m_B}(V - V_A)^2$$

$$\Rightarrow (V - V_A)(V + V_A) = \frac{m_A}{m_B}(V - V_A)^2$$

$$\Rightarrow V + V_A = \frac{m_A}{m_B}(V - V_A)$$

$$\Rightarrow \left(1 - \frac{m_A}{m_B}\right) V = -V_A - \frac{m_A}{m_B} V_A$$

$$\Rightarrow (m_B - m_A) V = -(m_B + m_A) V_A \quad \therefore \boxed{V_A = \left(\frac{m_A - m_B}{m_A + m_B}\right) V} \quad \text{V.}$$

Substitute V. into IV to find

$$V_B = \frac{m_A}{m_B}\left(V - \left(\frac{m_A - m_B}{m_A + m_B}\right) V\right) = \frac{m_A}{m_B}\left(\frac{\cancel{m_A} + m_B - \cancel{m_A} + m_B}{m_A + m_B}\right) V = \boxed{\left(\frac{2 m_A}{m_A + m_B}\right) V = V_B}$$

Notice that

$$V_B - V_A = \left(\frac{2m_A - m_A - m_B}{m_A + m_B}\right) V$$

$$\Rightarrow \boxed{V_B - V_A = -V}$$

Three-dimensional case

Again suppose the collision is elastic.

We have,

$$\text{I} \quad \tfrac{1}{2} m_A \vec{V}_{A1}\cdot\vec{V}_{A1} + \tfrac{1}{2} m_B \vec{V}_{B1}\cdot\vec{V}_{B1} = \tfrac{1}{2} m_A \vec{V}_{A2}\cdot\vec{V}_{A2} + \tfrac{1}{2} m_B \vec{V}_{B2}\cdot\vec{V}_{B2}$$

$$\text{II} \quad m_A \vec{V}_{A1} + m_B \vec{V}_{B1} = m_A \vec{V}_{A2} + m_B \vec{V}_{B2}$$

Note I yields

$$\vec{V}_{A1}\cdot\vec{V}_{A1} = \vec{V}_{A2}\cdot\vec{V}_{A2} + \frac{m_B}{m_A}\left(\vec{V}_{B2}\cdot\vec{V}_{B2} - \vec{V}_{B1}\cdot\vec{V}_{B1}\right)$$

$$\Rightarrow m_A\left(\vec{V}_{A1}\cdot\vec{V}_{A1} - \vec{V}_{A2}\cdot\vec{V}_{A2}\right) = m_B\left(\vec{V}_{B2}\cdot\vec{V}_{B2} - \vec{V}_{B1}\cdot\vec{V}_{B1}\right)$$

$$\Rightarrow \underline{m_A\left(\vec{V}_{A1} - \vec{V}_{A2}\right)\cdot\left(\vec{V}_{A1} + \vec{V}_{A2}\right) = m_B\left(\vec{V}_{B2} - \vec{V}_{B1}\right)\cdot\left(\vec{V}_{B2} + \vec{V}_{B1}\right)}. \quad \text{III}$$

Likewise, II yields,

$$\underline{m_A\left(\vec{V}_{A1} - \vec{V}_{A2}\right) = -m_B\left(\vec{V}_{B1} - \vec{V}_{B2}\right)} \quad \text{IV}$$

Substitute into III to obtain,

$$-m_B\left(\vec{V}_{B1} - \vec{V}_{B2}\right)\cdot\left(\vec{V}_{A1} + \vec{V}_{A2}\right) = m_B\left(\vec{V}_{B2} - \vec{V}_{B1}\right)\cdot\left(\vec{V}_{B2} + \vec{V}_{B1}\right) \quad \text{V}$$

jump!

$$\Rightarrow \vec{V}_{A1} + \vec{V}_{A2} = \vec{V}_{B2} + \vec{V}_{B1}$$

$$\Rightarrow \boxed{\vec{V}_{B1} - \vec{V}_{A1} = \vec{V}_{B2} - \vec{V}_{A2}}$$

$\therefore$ relative velocity constant under elastic collisions.

<u>Remark</u>: the math has a giant hole in it at the "jump". Generally $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$ does not imply $\vec{b} = \vec{c}$. If $\vec{V}_{B1} - \vec{V}_{B2} = \alpha\hat{k}$ then we only learn that $(\vec{V}_{A1} + \vec{V}_{A2})_z = (\vec{V}_{B2} + \vec{V}_{B1})_z$ and no info about the x,y-components is given by V.