EXAMPLES OF SERIES

A series is an infinite sum. A series may or may not converge. Let us give a precise definition

Def^n/ the series $S = a_1 + a_2 + \cdots = \sum_{k=1}^{\infty} a_k$ converges if $\lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k\right)$ converges. We say $S_n = \sum_{k=1}^{n} a_k$ is the n^{th}-partial sum of S. If the sequence $\{S_n\}_{n=1}^{\infty}$ of partial sums does not converge then $S = a_1 + a_2 + \cdots = \sum_{k=1}^{\infty} a_k$ is said to diverge.

Remark: when $S_n \to \pm\infty$ we also write $\sum_{k=1}^{\infty} a_k = \pm\infty$. We say $a_1 + a_2 + \cdots$ diverges to $\pm\infty$ in such cases.

Remark: some people prefer to say $S = a_1 + a_2 + \cdots$ is summable when the $\underbrace{\text{series converges}}_{\text{this means the sequence of partial sums converges.}}$

1.) $S = \sum_{k=1}^{\infty} 1 = 1 + 1 + \cdots$ (this series is a simple example)

$$S_n = \sum_{k=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n\text{-summands}} = n$$

Observe $\lim_{n\to\infty}(S_n) = \lim_{n\to\infty}(n) = \infty$

Thus $\boxed{\sum_{k=1}^{\infty} 1 = \infty}$ (divergent series)

2.) $S = \sum_{k=1}^{\infty} \left(\tan^{-1}(k+2) - \tan^{-1}(k) \right)$

CONVERGE? DIVERGE? If conv. to what does this series sum to?

$$S_n = \sum_{k=1}^{n} \left[\tan^{-1}(k+2) - \tan^{-1}(k) \right]$$

$$= \underbrace{\left(\cancel{\tan^{-1}(3)} - \tan^{-1}(1)\right)}_{k=1} + \underbrace{\left(\cancel{\tan^{-1}(4)} - \tan^{-1}(2)\right)}_{k=2} + \underbrace{\left(\cancel{\tan^{-1}(5)} - \cancel{\tan^{-1}(3)}\right)}_{k=3}$$

$$+ \cdots + \underbrace{\left(\cancel{\tan^{-1}(n)} - \cancel{\tan^{-1}(n-2)}\right)}_{k=n-2} + \underbrace{\left(\tan^{-1}(n+1) - \cancel{\tan^{-1}(n-1)}\right)}_{k=n-1} +$$

$$+ \underbrace{\left(\tan^{-1}(n+2) - \cancel{\tan^{-1}(n)}\right)}_{k=n}$$

$$= \tan^{-1}(n+2) + \tan^{-1}(n+1) - \tan^{-1}(1) - \tan^{-1}(2)$$

the n^{th} partial sum collapsed to these terms. Such series are said to be <u>telescoping</u>. Examples where this calculation is possible are unusual. But, nice,

$$\lim_{n\to\infty} (S_n) = \lim_{n\to\infty} \left(\underbrace{\tan^{-1}(n+2)}_{\frac{\pi}{2}} + \underbrace{\tan^{-1}(n+1)}_{\frac{\pi}{2}} - \underbrace{\tan^{-1}(1)}_{\frac{\pi}{4}} - \tan^{-1}(2) \right)$$

$$\boxed{S = \frac{5\pi}{4} - \tan^{-1}(2)}$$

3.) <u>GEOMETRIC</u> <u>SERIES</u> : $S = a + ar + ar^2 + \cdots$

a, r given real #'s

$$S_n = \sum_{k=1}^{n} ar^{k-1} = a + ar + \cdots + ar^{n-1}$$

$$r S_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Consequently, most terms cancel in the difference below,

$$S_n - r S_n = a - ar^n$$

If $r \neq 1$ then calculate $S_n - r S_n = (1-r) S_n$ and

$$S_n = \frac{a - ar^n}{1-r}$$

Suppose $|r| < 1$ then $r^n \longrightarrow 0$ as $n \longrightarrow \infty$ thus,

$$\lim_{n \to \infty} \left(\frac{a - ar^n}{1-r} \right) = \frac{a}{1-r}$$

Thus, $\boxed{\text{if } |r| < 1 \text{ then } \sum_{k=1}^{\infty} ar^{k-1} = a + ar + \cdots = \frac{a}{1-r}}$

4.) $0.99999.... = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$

$$= \frac{9}{10} + \frac{9}{10}\left(\frac{1}{10}\right) + \frac{9}{10}\left(\frac{1}{10}\right)^2 + \cdots$$

identify geom. $a = 9/10$, $r = 1/10$

$$= \frac{a}{1-r}$$

$$= \frac{9/10}{1 - 1/10}$$

$$= \frac{9/10}{9/10}$$

$$= \boxed{1.}$$

Decimal Rep. of real #'s already <u>assumes</u> existence and utility of series. You've been working with series w/o knowing it ☺

5.) $6.17171717\ldots = 6 + 0.\overline{17}$

Notice $0.\overline{17} = 0.17 + 0.0017 + 0.000017 + \cdots$

$$= \underbrace{0.17 + \frac{1}{100}(0.17) + \frac{1}{100^2}(0.17) + \cdots}_{a = 0.17 \text{ and } r = \frac{1}{100}}$$

$$= \frac{0.17}{1 - \frac{1}{100}}$$

$$= \frac{17/100}{99/100}$$

$$= \frac{17}{19} \quad \therefore \quad \underline{6.1717\ldots = 6 + \frac{17}{19}}.$$

Thm/ Any real # with infinitely repeating decimal expansion is a <u>rational</u> # $(\underbrace{\mathbb{Q} = \{ \frac{p}{q} \mid p, q \in \mathbb{Z},\ q \neq 0 \}}_{\text{RATIONAL \#'s}})$.

If $r \in \mathbb{R}$ and $r \notin \mathbb{Q}$ then r is <u>irrational</u>. For example, π, & e are irrational and hence have non-repeating decimal expansions (proving $\pi, e \notin \mathbb{Q}$ is way harder than most of what we do here...)

6.) $$\sum_{k=1}^{\infty} \left(3^k/2^{2k+6}\right) = \sum_{k=1}^{\infty} \frac{3^k}{2^{2k+6}}$$ (Sorry, messy pen here)

$$= \sum_{k=1}^{\infty} \frac{1}{2^6}\left(\frac{3}{4}\right)^k \quad : \quad 2^{2k} = (2^2)^k = 4^k.$$

geometric series with $a = 1/2^6$ & $r = 3/4$

$$= \frac{1/2^6}{1 - 3/4} = \boxed{\frac{1}{16}}$$

7.) $S = \displaystyle\sum_{k=3}^{\infty} \frac{2}{(k-1)(k-2)} =$

$$S_n = \sum_{k=3}^{n} \frac{2}{(k-1)(k-2)}$$ ← simplifies if we use partial fractions and study the collapse.

$$\frac{2}{(k-1)(k-2)} = \frac{A}{k-1} + \frac{B}{k-2}$$

$$2 = A(k-2) + B(k-1)$$

$k=1$ | $2 = A(1-2)$ $\therefore A = -2$

$k=2$ | $2 = B(2-1)$ $\therefore B = 2$

$$S_n = \sum_{k=3}^{n} \left(\frac{2}{k-2} - \frac{2}{k-1} \right)$$: factor out 2.

$$= 2\left[\left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \cdots + \left(\cancel{\tfrac{1}{n-3}} - \cancel{\tfrac{1}{n-2}}\right) + \left(\cancel{\tfrac{1}{n-2}} - \tfrac{1}{n-1}\right)\right]$$

$$= 2\left(1 - \frac{1}{n-1}\right)$$

Thus,

$$\sum_{k=3}^{\infty} \frac{2}{(k-1)(k-2)} = \lim_{n\to\infty} \left(\sum_{k=3}^{n} \frac{2}{(k-1)(k-2)} \right)$$

$$= \lim_{n\to\infty} \left(2\left(1 - \underbrace{\frac{1}{n-1}}_{0}\right)\right)$$

$$= \boxed{2.}$$

Remark: examples like this are rather special. Usually we are only able to determine convergence or divergence of the series. Here we could find the value of the series.

n^{th} TERM TEST (OR k^{th} term test etc...)

Suppose $\sum_{k=1}^{\infty} a_k$ converges. This means $\left\{\sum_{k=1}^{n} a_k\right\}_{n=1}^{\infty}$ is a convergent sequence (of partial sums). Notice,

$$\underbrace{\sum_{k=1}^{n} a_k}_{S_n} - \underbrace{\sum_{k=1}^{n-1} a_k}_{S_{n-1}} = a_n + \cancel{\sum_{k=1}^{n-1} a_k} - \cancel{\sum_{k=1}^{n-1} a_k} = a_n$$

Observe both S_n and S_{n-1} converge to $\sum_{n=1}^{\infty} a_n = S$

So, to summarize,

$$\lim_{n\to\infty} (a_n) = \lim_{n\to\infty} \left(S_n - S_{n-1} \right) = S - S = 0.$$

$$\boxed{\text{Th}^{m}/ \text{ If } \sum_{k=1}^{\infty} a_k \text{ converges then } \lim_{n\to\infty} (a_n) = 0}$$

- it follows, if $\lim_{n\to\infty} (a_n) \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges,

<u>Notation</u>: $\sum_{k=1}^{\infty} a_k = \sum_{n=1}^{\infty} a_n$ we can use whatever index we like for the index of summation. I've mostly used k upto this point to keep it simple, but I'll take off the training wheels now ...

8.) $\sum_{n=1}^{\infty} ar^{n-1}$ where $|r| \geq 1$ has $a_n = ar^{n-1}$ and $\lim_{n\to\infty} (ar^{n-1}) \neq 0 \therefore \sum_{n=1}^{\infty} ar^{n-1}$ diverges in case $|r| \geq 1$.

9.) it can be shown that $\underbrace{\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots}_{\text{"HARMONIC" SERIES}}$

diverges. Notice

$a_n = \frac{1}{n} \longrightarrow 0$ as $n \to \infty$, This does

NOT disagree with the n^{th} term test.

$$\sum_{n=1}^{\infty} a_n \text{ CONVERGES} \Longrightarrow \lim_{n \to \infty} (a_n) = 0$$
$$\not\Longleftarrow$$

Th^m/ If $\sum_1 a_n$, $\sum_1 b_n$ are convergent then

$$\sum_1 (a_n \pm b_n) = \sum_1 a_n \pm \sum_1 b_n$$
$$\sum_1 (c a_n) = c \sum_1 a_n \quad \text{for some constant } c.$$

10.) $\displaystyle \sum_{n=0}^{\infty} \frac{3^{n+2} - 5^n}{6^n} = \sum_{n=0}^{\infty} \frac{3^{n+2}}{6^n} - \sum_{n=0}^{\infty} \frac{5^n}{6^n}$ $\Big\}$ ✱

$$= 9 \sum_{n=0}^{\infty} \left(\frac{3}{6}\right)^n - \sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n$$

$$= 9 \Big(\underbrace{1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \dots}_{\substack{a=1 \\ r=1/2}}\Big) - \Big(\underbrace{1 + \tfrac{5}{6} + \left(\tfrac{5}{6}\right)^2 + \dots}_{\substack{a=1 \\ r=5/6}}\Big)$$

$$= 9\left(\frac{1}{1 - 1/2}\right) - \frac{1}{1 - 5/6}$$

$$= 18 - 6$$

$$= \boxed{12.}$$

✱: these steps are only reasonable since the resulting sums are convergent.