

<It is clear that $[\bar{X}, \bar{Y}]$ is a linear mapping from $C_{loc}^\infty(x)$ to \mathbb{R} , since $\bar{X}(f)$ and $\bar{Y}(f)$ are in $C_{loc}^\infty(M)$ for $f \in C_{loc}^\infty(M)$ (recall that $\bar{X}(f)(x) = \bar{X}_x(f)$ is smooth since \bar{X} and f are smooth).

Since \bar{X} is smooth clearly $x \mapsto \bar{X}_x(\bar{Y}(f))$ is smooth for smooth f . Similarly $x \mapsto \bar{Y}_x(\bar{X}(f))$ is smooth for smooth f . So if we show $[\bar{X}, \bar{Y}]$ has the Leibnitz property then it will follow that it is a tangent vector for each $x \in M$ and it will also follow from the above remarks that

$$x \mapsto [\bar{X}, \bar{Y}]_x(f)$$

is smooth for smooth f and thus that $[\bar{X}, \bar{Y}]$ is a smooth vector field. Let $x \in M$, $f, g \in C_{loc}^\infty(x)$ and observe that

$$\begin{aligned} [\bar{X}, \bar{Y}]_x(fg) &= \bar{X}_x(\bar{Y}(fg)) - \bar{Y}_x(\bar{X}(fg)) \\ &= \bar{X}_x(f \bar{Y}(g) + g \bar{Y}(f)) - \bar{Y}_x(f \bar{X}(g) + g \bar{X}(f)) \\ &= f(x) \bar{X}_x(\bar{Y}(g)) + \cancel{\bar{Y}(g) \bar{X}_x(f)} + g(x) \bar{X}_x(\bar{Y}(f)) \\ &\quad + \cancel{\bar{X}_x(g) \bar{Y}_x(f)} - f(x) \bar{Y}_x(\bar{X}(g)) - \cancel{\bar{Y}_x(f) \bar{X}_x(g)} \\ &\quad - g(x) \bar{Y}_x(\bar{X}(f)) - \cancel{\bar{Y}_x(g) \bar{X}_x(f)} \\ &= f(x) [\bar{X}, \bar{Y}](g) + g(x) [\bar{X}, \bar{Y}](f). \end{aligned}$$

So $[\bar{X}, \bar{Y}] \in \Gamma(M)$ for $\bar{X}, \bar{Y} \in \Gamma(M)$.

Definition A Lie algebra is a vector space $(L, +, \circ)$ on which there is defined a bilinear mapping from $L \times L$ to L such that for $x, y \in L$ the value of the mapping at (x, y) is denoted $[x, y]$ and in addition to the bilinearity in x and y one also has the properties:

- $[x, y] = -[y, x]$
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$. When (i) is satisfied we say that the operation $[\cdot, \cdot]$ is skew-symmetric and we refer to (ii) as the Jacobi identity.

Remark: Note that generally a Lie algebra operation $[\cdot, \cdot]$ is not associative and in fact the Jacobi identity is a "replacement" for the associative property.

Example 1 Let $L = \mathbb{R}^3$ with the usual operations $+$ and \circ . Define $[a, b] = axb$ for $a, b \in \mathbb{R}^3$. Then L is a Lie algebra with this definition of the Lie operation $[\cdot, \cdot]$.

Example 2 Let $L = gl(n)$ with the usual operations $+$ and \circ . Let $[\cdot, \cdot]$ be defined on $gl(n)$ by $[A, B] = AB - BA$ for $A, B \in gl(n)$. Then L is a Lie algebra relative to these operations. Notice that if \mathcal{J} is any subspace of $gl(n)$ such that $A, B \in \mathcal{J}$ implies $AB - BA \in \mathcal{J}$ then \mathcal{J} will also be a Lie algebra and \mathcal{J} is

referred to as a Lie subalgebra of $\mathfrak{gl}(n)$.
The reader should verify that both \mathfrak{g}_J and $s(\mathfrak{g}_J)$ are Lie subalgebras of $\mathfrak{gl}(n)$.

Example 3 Let $L = \Gamma(M)$ the set of all vector fields on a manifold M . Recall the operations on $\Gamma(M)$ defined by

$$(X+Y)_x = X_x + Y_x \quad (\text{sum of tangent vectors})$$

$$(cX)_x = c X_c \quad (\text{scalar multiple of tangent vector})$$

$$[X, Y]_x = X_x \circ Y - Y_x \circ X \quad (\text{Lie bracket operation})$$

It is routine to verify that $\Gamma(M)$ is a Lie algebra relative to these operations though the Jacobi identity may prove to be tedious to verify.

Definition If $(L, +, \cdot, [\cdot, \cdot])$ is a Lie algebra and $L_0 \subset L$ is a vector subspace of L such that for $x, y \in L_0$ it follows that $[x, y] \in L_0$ then we say L_0 is a sub-Lie algebra of L . Clearly it is a new Lie algebra under the restrictions of the operations of L to L_0 .

Theorem If G is a Lie group then $T_{\text{inv}}(G)$ is a sub-Lie algebra of the Lie algebra of all vector fields on G .

Before getting into the proof of the theorem we need a definition and a lemma.

Definition Let M and N be manifolds and $\varphi: M \rightarrow N$ a smooth mapping. If $X \in \Gamma(M)$ and $Y \in \Gamma(N)$ then we say X and Y are φ -related iff for each $x \in X$

$$d_x \varphi(X_x) = Y_{\varphi(x)}.$$

Notice that we may also write this condition as $d\varphi \circ X = Y \circ \varphi$.

Lemma Let M and N be manifolds and $\varphi: M \rightarrow N$ a smooth mapping. Assume that $X_1, X_2 \in \Gamma(M)$ and $Y_1, Y_2 \in \Gamma(N)$. If X_i is φ -related to Y_i for $i=1,2$ then $[X_1, X_2]$ is φ -related to $[Y_1, Y_2]$.

Proof of the Lemma. Since X_i is φ -related to Y_i for $i=1,2$ we see that

$$[Y_i(g) \circ \varphi](x) = (Y_i)_{\varphi(x)}(g) = d_x \varphi(X_i)(g) = (X_i)_x(g \circ \varphi) = X_i(g \circ \varphi)(x)$$

for $x \in M$, $g \in C_{loc}^\infty(\varphi(x))$. It follows that

$$\begin{aligned} d\varphi([X_1, X_2]_x)(g) &= [X_1, X_2]_x(g \circ \varphi) = (X_1)_x(X_2(g \circ \varphi)) - (X_2)_x(X_1(g \circ \varphi)) \\ &= (X_1)_x(Y_2(g) \circ \varphi) - (X_2)_x(Y_1(g) \circ \varphi) \\ &= (Y_1)_{\varphi(x)}(Y_2(g)) - (Y_2)_{\varphi(x)}(Y_1(g)). \end{aligned}$$

and so

$$d\varphi([\bar{X}_1, \bar{X}_2]_x) = [\bar{Y}_1, \bar{Y}_2]_{\varphi(x)}$$

for each $x \in M$. Thus $[\bar{X}_1, \bar{X}_2]$ is φ -related to $[\bar{Y}_1, \bar{Y}_2]$ as asserted in the lemma.

Proof of the Theorem. Notice that if G is a Lie group then $X \in \Gamma(G)$ is left-invariant iff

$$d_x l_a(X_x) = X(l_a(x))$$

for all $a, x \in G$ and this is equivalent to saying

that $d_x l_a(X_x) = X_{ax}$ for all $x, a \in G$. Thus

X is left-invariant iff X is l_a -related to X for all $a \in G$. It follows that if \bar{X}, \bar{Y} are left-invariant vector fields on G then

\bar{X} is l_a -related to X and \bar{Y} is l_a -related to Y

for all $a \in G$ and consequently, by the lemma,

$[\bar{X}, \bar{Y}]$ is l_a -related to $[X, Y]$ for all a .

It follows that $[\bar{X}, \bar{Y}] \in \Gamma_{inv}(G)$ whenever $\bar{X}, \bar{Y} \in \Gamma_{inv}(G)$.

So $\Gamma_{inv}(G)$ is closed under the bracket $[\cdot, \cdot]$

defined on $\Gamma(G)$. It is trivial to show

that if $\bar{X}, \bar{Y} \in \Gamma_{inv}(G)$ then $\bar{X} + \bar{Y} \in \Gamma_{inv}(G)$

as

$$d_x l_a((\bar{X} + \bar{Y})_x) = d_x l_a(\bar{X}_x) + d_x l_a(\bar{Y}_x)$$

$$= \bar{X}_{ax} + \bar{Y}_{ax} = (\bar{X} + \bar{Y})_{ax}$$

for all $a, x \in G$. Similarly $c \in \mathbb{R}, \bar{X} \in \Gamma_{inv}(G)$ imply that $c\bar{X} \in \Gamma_{inv}(G)$. It follows that $\Gamma_{inv}(G)$ is a Lie subalgebra of $\Gamma(G)$. \square

Notice that $\Gamma(M)$ is generally a vector space, a Lie-algebra and a $C^\infty(M)$ module where the module operation is defined by $(f \cdot X)_x = f(x)X_x$ for $f \in C^\infty(M)$, $X \in \Gamma(M)$. Moreover, locally

$$X = a^i \frac{\partial}{\partial x^i}$$

in terms of a chart (x^i) . Since the a^i are smooth functions defined on the domain U of (x^i) and since $\frac{\partial}{\partial x^i}$ are also defined on U we see that, as a module, $\Gamma(U)$ is finitely generated. Thus $\Gamma(M)$ is locally finitely generated but as a vector space is infinite dimensional.

On the other hand $\Gamma_{\text{inv}}(G)$ is a finite dimensional vector space but is not a submodule of $\Gamma(G)$. Generally if X is left-invariant then fX will not be left-invariant.

Recall that $gl(n)$ is a vector space and so is a manifold with global chart the identity map. Moreover the identity on $Gl(n)$ maps $Gl(n)$ onto an open subset of the vector space $gl(n)$ as $Gl(n)$ is open in $gl(n)$. Define real-valued functions $x_{ij}: Gl(n) \rightarrow \mathbb{R}$ by $x_{ij}(A)$ is the entry in the i -th row and j -th column of A . Clearly x_{ij} is smooth for all i, j and are the components of an admissible chart of $Gl(n)$.

Theorem The tangent space of $\mathrm{GL}(n)$ at the identity $I \in \mathrm{GL}(n)$ may be identified with $\mathrm{gl}(n)$. If $B \in \mathrm{gl}(n) = T_I \mathrm{GL}(n)$ then the left-invariant vector field X^B determined by B is given in coordinates by

$$X^B = \sum_{R,i,j} B_{Rj} x_{iR} \left(\frac{\partial}{\partial x_{ij}} \right).$$

Thus, for $A \in \mathrm{GL}(n)$

$$X^B(A) = \sum_{i,j} (AB)_{ij} \frac{\partial}{\partial x_{ij}}$$

Moreover, the Lie algebra $T_{\text{inv}}(\mathrm{GL}(n))$ is isomorphic to $(\mathrm{gl}(n), \cdot, [\cdot, \cdot])$ as defined in Example 2 above. In particular, for $B_1, B_2 \in \mathrm{gl}(n)$,

$$[X^{B_1}, X^{B_2}] = X^{[B_1, B_2]}$$

Proof Since $\mathrm{GL}(n)$ is open in the vector space $\mathrm{gl}(n)$ it is clear that $T_A(\mathrm{GL}(n))$ may be identified with $\mathrm{gl}(n)$ for each $A \in \mathrm{GL}(n)$. Under this identification $B \in \mathrm{gl}(n)$ is identified as the components of $(\frac{\partial}{\partial x_{ij}}|_A)$, i.e., B is identified with

$$B_{ij} \left(\frac{\partial}{\partial x_{ij}}|_A \right) \in T_A(\mathrm{GL}(n)).$$

Notice that if $f: \mathrm{GL}(n) \rightarrow \mathrm{GL}(n)$ is any smooth mapping then $d_f: T_A \mathrm{GL}(n) \rightarrow T_{f(A)} \mathrm{GL}(n)$

is locally defined in terms of the Fréchet derivative of a local coordinate representative of f . Since $\text{GL}(n)$ is open in $\text{gl}(n)$, we may choose the identity map as a chart and so a local representative of f is just f again. So df is given in terms of the Fréchet derivative of f . More precisely, if Df is the Fréchet derivative of f at $A \in \text{GL}(n)$, then

$$df_A\left(B_{ij}\left(\frac{\partial}{\partial x_{ij}}|_A\right)\right) = (Df)(B)_i j \left(\frac{\partial}{\partial x_{ij}}|_{f(A)}\right).$$

(Generally the components of any vector v in the domain of df are transformed to components of the image vector $df_A(v)$ via the Fréchet derivative Df_A). Since B_{ij} are the components of $B_{ij}\left(\frac{\partial}{\partial x_{ij}}\right)$ the components of $df_A\left(B_{ij}\left(\frac{\partial}{\partial x_{ij}}\right)\right)$ must be given by letting the Fréchet derivative Df_A operate on the components $B = (B_{ij})$ and so the components of $df_A\left(B_{ij}\left(\frac{\partial}{\partial x_{ij}}\right)\right)$ must be given by $(Df)(B)_{ij}$. If f is the restriction of a linear map from $\text{gl}(n)$ to $\text{gl}(n)$ then $Df_A = f$ and

$$df_A\left(B_{ij}\left(\frac{\partial}{\partial x_{ij}}|_A\right)\right) = Df(B)_{ij}\left(\frac{\partial}{\partial x_{ij}}|_{f(A)}\right) = f(B)_{ij}\left(\frac{\partial}{\partial x_{ij}}|_{f(A)}\right)$$

In particular, if $f = l_A$ for $A \in \text{GL}(n)$, then $\sum^B(A) = df_A(B)$

We now show that $[\bar{X}^{B_1}, \bar{X}^{B_2}] = \bar{X}^{[B_1, B_2]}$. Since $\bar{X}^{B_1}, \bar{X}^{B_2}$, $\bar{X}^{[B_1, B_2]}$ and $[\bar{X}^{B_1}, \bar{X}^{B_2}]$ are left-invariant vector fields they are uniquely determined by their value at the identity I of the group $GL(n)$. Thus it suffices to show that

$$[\bar{X}^{B_1}, \bar{X}^{B_2}]_I = \bar{X}^{[B_1, B_2]}_I.$$

Let $f \in C_{loc}^\infty(I)$ and observe that

$$\bar{X}_I^{B_1}(\bar{X}_I^{B_2}f) = \bar{X}_I^{B_1}\left(\sum_{i,j,k} B_{2kj} x_{ik} \frac{\partial f}{\partial x_{ij}}\right)$$

$$= \sum_{a,b} B_{1ab} \left(\frac{\partial}{\partial x_{ab}} \Big|_I \right) \left(\sum_{i,j,k} B_{2kj} x_{ik} \frac{\partial f}{\partial x_{ij}} \right)$$

$$= \sum_{a,b} B_{1ab} \sum_{i,j,k} B_{2kj} \frac{\partial x_{ik}}{\partial x_{ab}} \frac{\partial f}{\partial x_{ij}}$$

$$+ \sum_{a,b} \sum_{i,j,k} B_{1ab} B_{2kj} x_{ik}(I) \left(\frac{\partial^2 f}{\partial x_{ab} \partial x_{ij}} \right)$$

$$= \sum_{i,j,k} B_{1ik} B_{2kj} \frac{\partial f}{\partial x_{ij}}(I)$$

$$+ \sum_{i,j,a,b} B_{1ab} B_{2ij} \left(\frac{\partial^2 f}{\partial x_{ab} \partial x_{ij}} \right)$$

$$= \sum_{i,j} (B_1 B_2)_{ij} \left(\frac{\partial}{\partial x_{ij}} \Big|_I \right)(f)$$

$$+ \sum_{i,j,a,b} B_{1ab} B_{2ij} \frac{\partial^2 f}{\partial x_{ab} \partial x_{ij}}$$

It follows that

$$\bar{X}_I^{B_1}(\bar{X}_I^{B_2}(f)) - \bar{X}_I^{B_2}(\bar{X}_I^{B_1}(f))$$

$$= \sum_{i,j} (B_1 B_2)_{ij} \left(\frac{\partial}{\partial x_{ij}} \Big|_I \right) f - \sum_{i,j} (B_2 B_1)_{ij} \left(\frac{\partial}{\partial x_{ij}} \Big|_I \right) f$$

$$= \bar{X}_I^{[B_1, B_2]}(f)$$

and since f is arbitrary, $[\bar{X}_I^{B_1}, \bar{X}_I^{B_2}] = \bar{X}_I^{[B_1, B_2]}$.

The Theorem follows.

Definition If G is a Lie group then we say that the Lie algebra of left-invariant vector fields on G is the Lie algebra of G . We denote this Lie algebra by \mathfrak{g} or by $\Gamma_{\text{inv}}^l(G)$. Notice that since \mathfrak{g} is vector space isomorphic to $T_e G$ one may use the isomorphism to induce a Lie bracket on $T_e G$. One may then identify \mathfrak{g} with $T_e G$ and we do so without further comment.

Remark The last theorem shows that the Lie algebra of $\text{GL}(n)$ may be identified with $\mathfrak{gl}(n)$.

Definition Let G be a Lie group and H a subset of G . We say H is a Lie subgroup of G iff it is a Lie group such that

- (1) H is a subgroup of G , and
- (2) H is a submanifold of G .

Example Observe that G_J and $S(G_J)$ are Lie subgroups of $GL(n)$. Similarly G_J^C and $S(G_J^C)$ are Lie subgroups of $GL(n, \mathbb{C})$.

Remark. If G is a Lie group and H is a Lie subgroup of G then the tangent space of H at $x \in H$ may be identified as a subspace of the tangent space of G at x . Perhaps the easiest way to see this is to notice that if $i: H \hookrightarrow G$ is the inclusion mapping, $i(x) = x$ for all $x \in H$, then i is smooth (prove this!) and so $d_i: T_x H \rightarrow T_x G$ is a well-defined linear transformation. It is easy to show that d_i is injective so $T_x H$ is identified with $d_i(T_x H) \subset T_x G$. We know that every left invariant vector field on H has the form X_H^v where $X_H^v(x) = d_{\dot{x}} l_x^H(v)$ for some $v \in T_e H$ and for all $x \in H$. Here l_x^H denotes left translation by x in the group H . On the other hand $l_x^G \circ i = l_x^H$ where l_x^G is left translation in G by $x \in H$. Thus if $\dot{X}_G^{d_i(v)}$ is the left invariant vector field of G determined by $d_i(v) \in T_e G$ then, for $x \in H$,

$$\begin{aligned} \dot{X}_G^{d_i(v)}(x) &= d_{\dot{x}} l_x^G(d_i(v)) = d_{\dot{x}} (l_x^G \circ i)(v) = d_{\dot{x}} l_x^H(v) \\ &= X_H^v(x). \end{aligned}$$

So $\dot{X}_G^{d_i(v)} \circ i = X_H^v$ for every $v \in T_e H$, i.e., the restriction of $\dot{X}_G^{d_i(v)}$ to H is X_H^v .

It follows that each left-invariant vector field \underline{X}_H^v on H uniquely determines a left invariant vector field $\underline{X}_G^{di(v)}$ on G whose restriction to H is \underline{X}_H^v . Notice also that for each $v \in T_e H$, \underline{X}_H^v is i -related to $\underline{X}_G^{di(v)}$ in the sense of the lemma to Theorem since for $x \in H$

$$\begin{aligned} di_x(\underline{X}_H^v(x)) &= di_x(d\ell_x^H(v)) = d(i \circ \ell_x^H)(v) = d(\ell_x^G \circ i)(v) \\ &= d\ell_x^G(d_i(v)) = \underline{X}_G^{di(v)}(x) = \underline{X}_G^{di(v)}(i(x)), \end{aligned}$$

i.e. $di \circ \underline{X}_H^v = \underline{X}_G^{di(v)} \circ i$ for $v \in T_e H$.

Theorem Let G be a Lie group and H a Lie subgroup of G . Then the Lie algebra \mathfrak{h} of H may be identified as a Lie-subalgebra of the Lie algebra \mathfrak{g} of G . One identifies the left-invariant vector field \underline{X}_H^v of H determined by $v \in T_e H$ with the left-invariant vector field $\underline{X}_G^{di(v)}$ of G determined by $di(v) \in T_e G$.

Proof Notice that one has two linear injections

$\Phi: T_e H \rightarrow \Gamma_{inv}(H)$, $\Psi: T_e H \rightarrow \Gamma_{inv}(G)$ defined by

$\Phi(v) = \underline{X}_H^v$ and by $\Psi(v) = \underline{X}_G^{di(v)}$ respectively.

Moreover we know Φ is surjective as well.

Thus $\Psi \circ \Phi^{-1}$ is a linear injection of $\Gamma_{inv}(H)$ into $\Gamma_{inv}(G)$.

So the vector space $\Gamma_{inv}(H)$ is vector space isomorphic to a subspace of $\Gamma_{inv}(G)$. We show that $\Psi \circ \Phi^{-1}$ preserves the Lie algebra structures. We proved above that for $v \in T_e H$, $\Phi(v)$ is i -related to $\Psi(v)$. Thus for $v, w \in T_e H$,

$[\Phi(v), \Phi(w)]$ is 2-related to $[\bar{\Psi}(v), \bar{\Psi}(w)]$ (this follows from the lemma to Theorem). Since $[\Phi(v), \Phi(w)]$ is left invariant on H there exists a unique vector $u \in T_e H$ such that

$$[\Phi(v), \Phi(w)] = \Phi(u) \quad (\Phi \text{ is surjective}).$$

We have that $[\Phi(v), \Phi(w)] = \Phi(u)$ is 2-related to $\bar{\Psi}(u)$. So $\Phi(u)$ is 2-related to both $\bar{\Psi}(u)$ and $[\bar{\Psi}(v), \bar{\Psi}(w)]$. Thus $[\bar{\Psi}(v), \bar{\Psi}(w)]$ and $\bar{\Psi}(u)$ are left invariant vector fields on H such that

$$[\bar{\Psi}(v), \bar{\Psi}(w)] \circ i = d_i \circ \bar{\Phi}(u) = \bar{\Psi}^*(u) \circ i.$$

Since they agree at $e \in H$, it follows that

$$[\bar{\Psi}(v), \bar{\Psi}(w)] = \bar{\Psi}(u),$$

so

$$(\bar{\Psi} \circ \bar{\Phi}^{-1})([\bar{\Phi}(v), \bar{\Phi}(w)]) = \bar{\Psi}(\bar{\Phi}^{-1}(\bar{\Phi}(u)))$$

$$= \bar{\Psi}(u) = [\bar{\Psi}(v), \bar{\Psi}(w)]$$

$$= [(\bar{\Psi} \circ \bar{\Phi}^{-1})(\bar{\Phi}(v)), (\bar{\Psi} \circ \bar{\Phi}^{-1})(\bar{\Phi}(w))].$$

Since Φ is surjective,

$$(\bar{\Psi} \circ \bar{\Phi}^{-1})([\bar{\Phi}(v), \bar{\Phi}(w)]) = [(\bar{\Psi} \circ \bar{\Phi}^{-1})(\bar{\Phi}(v)), (\bar{\Psi} \circ \bar{\Phi}^{-1})(\bar{\Phi}(w))]$$

for all $v, w \in \Gamma_{\text{inv}}(H)$. The theorem follows.

Let G be a Lie sub-group of $\text{GL}(n)$.

Then $T_I G \subseteq T_I \text{GL}(n)$ and since $T_I \text{GL}(n)$ may be identified with $\text{gl}(n)$ we see that $T_I G$ may be identified with a subspace of $\text{gl}(n)$.

Indeed we have

$$T_I G = \left\{ A \in \text{gl}(n) \mid A = \gamma'(0) \text{ for some curve } \gamma: (a, a) \rightarrow G \ni \gamma(0) = I \right\}$$

Now the Lie algebra \mathfrak{g} of G is $\{\mathbb{X}^B \mid B \in T_I G\}$.

Moreover we know that

$$[\mathbb{X}^{B_1}, \mathbb{X}^{B_2}] = \mathbb{X}^{[B_1, B_2]}$$

for all $B_1, B_2 \in \text{gl}(n)$ and so in particular this holds for $B_1, B_2 \in T_I G \subseteq \text{gl}(n)$. Since G is a Lie subgroup of $\text{GL}(n)$, \mathfrak{g} is a Lie sub-algebra of the Lie algebra

$$\{\mathbb{X}^B \mid B \in \text{gl}(n)\}.$$

Thus $B_1, B_2 \in T_I G \Rightarrow [B_1, B_2] \in T_I G$.

Corollary If G is a Lie subgroup of $\text{GL}(n)$ then $T_I G \subseteq \text{gl}(n)$ is a Lie subalgebra of $\text{gl}(n)$ which is isomorphic as a Lie algebra to $T_{\text{inv}}(G)$.

The reader will have noticed that to each Lie group G we have assigned a Lie algebra, namely the Lie algebra of all left-invariant vector fields on G . Moreover, this Lie algebra, $\Gamma_{\text{inv}}(G)$, is isomorphic as a vector space to $T_e G$. The mapping Φ from $T_e G$ into $\Gamma_{\text{inv}}(G)$ defined by $\Phi(v) = \bar{X}^v$ is such an isomorphism. Using this isomorphism we can define a Lie structure on $T_e G$ by forcing Φ to be a Lie algebra isomorphism.

Thus if $v, w \in T_e G$ we define

$$[v, w] = \Phi^{-1}([\Phi(v), \Phi(w)]) = \Phi([X^v, X^w]).$$

The reader may verify that the mapping from $T_e G \times T_e G$ to $T_e G$ defined by

$$(v, w) \mapsto [v, w]$$

is indeed bilinear, skew-symmetric and satisfies the Jacobi identity.

Definition If G is any Lie group we denote its Lie algebra by $\mathfrak{l} = \mathfrak{l}(G)$. This Lie algebra $\mathfrak{l}(G)$ will denote either $(\Gamma_{\text{inv}}(G), +, \cdot, [\cdot, \cdot])$ or the isomorphic Lie algebra $(T_e G, +, \cdot, [\cdot, \cdot])$. If G and H are arbitrary Lie groups and $\varphi: G \rightarrow H$ is any smooth mapping which is also a homomorphism we denote by $\mathfrak{l}(\varphi)$ either the mapping $d\varphi: T_e G \rightarrow T_{\varphi(e)} H$ or the induced mapping $\tilde{\varphi}: \Gamma_{\text{inv}}(G) \rightarrow \Gamma_{\text{inv}}(H)$ defined by $\tilde{\varphi}(\bar{X}_G^v) = \bar{X}_{\varphi(e)}^{d\varphi(v)}$. We will regard both these mappings as functions from the Lie algebra $\mathfrak{l}(G)$ to the Lie algebra $\mathfrak{l}(H)$.

Remark One often refers to the category \mathcal{G} of Lie groups. The objects of the category is the class of all Lie groups. The so-called morphisms of the category are Lie group homomorphisms, i.e., smooth mappings from one Lie group to another which are also group homomorphisms. One also speaks of the category \mathcal{L} of Lie algebras. The objects of \mathcal{L} is the class of all Lie algebras and its morphisms are homomorphisms from one Lie algebra to another. The function l introduced in the last definition is a mapping from the category \mathcal{G} to the category \mathcal{L} . It assigns to each object (Lie group) of \mathcal{G} an object (Lie algebra) of \mathcal{L} and it assigns to each morphism (Lie group homomorphism) of \mathcal{G} a morphism (Lie algebra homomorphism) of \mathcal{L} . Moreover the reader may easily show that if $\varphi: G \rightarrow H$ and $\psi: H \rightarrow K$ are Lie group homomorphisms then $\psi \circ \varphi: G \rightarrow K$ is a Lie group homomorphism and one has Lie algebra homomorphisms $l(\varphi): l(G) \rightarrow l(H)$, $l(\psi): l(H) \rightarrow l(K)$, $l(\psi \circ \varphi): l(G) \rightarrow l(K)$. Moreover one has

$$l(\psi \circ \varphi) = l(\psi) \circ l(\varphi)$$

for all such φ, ψ . Thus l is a functor from the category \mathcal{G} into the category \mathcal{L} .

Theorem If G and H are Lie groups and $\varphi: G \rightarrow H$ is a Lie group homomorphism, then the mapping $l(\varphi): l(G) \rightarrow l(H)$ is a Lie algebra homomorphism. Moreover if $\varphi: G \rightarrow H$ and $\psi: H \rightarrow K$ are both Lie group homomorphisms, then $l(\psi \circ \varphi) = l(H) \circ l(\varphi)$.

Proof First note that $d\varphi: T_e G \rightarrow T_e H$ is a linear mapping as is also the induced mapping $\tilde{\varphi}: \Gamma_{\text{inv}}(G) \rightarrow \Gamma_{\text{inv}}(H)$ defined by

$$\tilde{\varphi}(X_G^v) = \sum_H \frac{d\varphi(v)}{H}$$

for $v \in T_e G$. We show that $\tilde{\varphi}$ preserves the Lie bracket of $\Gamma_{\text{inv}}(G)$. First notice that if $a, x \in G$ then

$$(\varphi \circ l_a^G)(x) = \varphi(ax) = \varphi(a)\varphi(x) = (l_{\varphi(a)}^H \circ \varphi)(x)$$

and consequently for each $x \in G$,

$$\begin{aligned} d\varphi(X_G^v) &= d\varphi(d l_x^G v) \\ &= d(\varphi \circ l_x^G)(v) \\ &= d(l_{\varphi(x)}^H \circ \varphi)(v) \\ &= d l_{\varphi(x)}^H(d\varphi(v)) \\ &= \sum_H \frac{d\varphi(v)}{H}(x). \end{aligned}$$

Thus X_G^v is φ -related to $X_H^{d\varphi(v)}$ for each $v \in T_e G$.

By the lemma to Theorem we see that for

$v, w \in T_e G$, $[X_G^v, X_G^w]$ is φ -related to

$[X_H^{d\varphi(v)}, X_H^{d\varphi(w)}]$. It follows that

$$d\varphi(\bar{X}_G^v, \bar{X}_G^w) = [\bar{X}_H^{d\varphi(v)}, \bar{X}_H^{d\varphi(w)}]_{(e)}$$

or

$$d\varphi(\bar{X}_G^{[v,w]}_{(e)}) = \bar{X}_H^{[d\varphi(v), d\varphi(w)]} \quad (e)$$

or

$$d\varphi([v,w]) = [d\varphi(v), d\varphi(w)].$$

Thus

$$l(\varphi)([v,w]) = [l(\varphi)(v), l(\varphi)(w)]$$

for $v, w \in T_e G$. For the sake of clarity we

also observe that

$$\begin{aligned} \widehat{\varphi}(\bar{X}_G^v, \bar{X}_G^w) &= \widehat{\varphi}(\bar{X}_G^{[v,w]}) \\ &= \bar{X}_H^{d\varphi([v,w])} \\ &= \bar{X}_H^{[d\varphi(v), d\varphi(w)]} \\ &= [\bar{X}_H^{d\varphi(v)}, \bar{X}_H^{d\varphi(w)}] \end{aligned}$$

and so $\widehat{\varphi}$ is indeed a Lie-algebra homomorphism as it should be if our identifications are to be consistent.Finally observe that $d(\psi \circ \varphi) = d\psi \circ d\varphi$ and also

$$\begin{aligned} \widetilde{\psi \circ \varphi}(\bar{X}_G^v) &= \bar{X}_K^{d(\psi \circ \varphi)(v)} = \bar{X}_K^{d\psi(d\varphi(v))} \\ &= \widetilde{\psi}(\bar{X}_H^{d\varphi(v)}) = (\widetilde{\psi} \circ \widetilde{\varphi})(\bar{X}_G^v) \end{aligned}$$

for all $v \in T_e G$. Thus we have both $d(\psi \circ \varphi) = d\psi \circ d\varphi$ and $\widetilde{\psi \circ \varphi} = \widetilde{\psi} \circ \widetilde{\varphi}$ and consequently $l(\psi \circ \varphi) = l(\psi) \circ l(\varphi)$.

Remark Observe that if $\varphi: G \rightarrow G$ is the identity then $l(\varphi): l(G) \rightarrow l(G)$ is the identity on $l(G)$. Moreover, if $\varphi: G \rightarrow H$ and $\psi: H \rightarrow G$

are Lie group homomorphisms which are inverses of one another so that

$$\psi \circ \varphi = i_G \quad \varphi \circ \psi = i_H$$

then

$$l(\psi) \circ l(\varphi) = i_{l(G)} \quad l(\varphi) \circ l(\psi) = i_{l(H)}$$

and consequently $l(\psi)$, $l(\varphi)$ are Lie algebra homomorphisms which are inverses of one another. Thus Lie group isomorphisms are mapped by l to Lie algebra isomorphisms. In particular

$$l(\varphi^{-1}) = l(\varphi)^{-1}$$

for each such φ .

Remark. We have seen that each Lie group G gives rise to a Lie algebra $l(G)$. This correspondence is not bijective since one can have $l(G) \cong l(H)$ for Lie groups G, H which are not Lie group isomorphic. Although it is beyond the scope of this manuscript to prove it, it is true that if G, H are simply connected Lie groups and $l(G) \cong l(H)$ then G and H are Lie group isomorphic. Moreover if G is simply connected and H is any Lie group such that $l(G) \cong l(H)$ then H is Lie group isomorphic to G/F where F is a finite normal sub-group of G . Generally G/F will not be simply connected. These theorems provide a valuable tool for deeper results in Lie theory.

Recall that a p -form ω on a manifold M is a smooth mapping from M into $\bigwedge^p M = \bigcup_{x \in M} \bigwedge^p(T_x M)$ such that $\omega(\bigwedge^p(T_x M)) = x$. Thus, for $x \in M$, $\omega(x) = \omega_x$ is an alternating multilinear mapping from $T_x M \times T_x M \times \dots \times T_x M$ (p -factors) into \mathbb{R} . Moreover ω is smooth at x_0 iff for every choice of smooth vector fields X_1, X_2, \dots, X_p defined on an open subset U of M containing x_0 , the mapping

$$x \mapsto \omega_x(X_1(x), \dots, X_p(x))$$

is smooth on U .

Definition If G is a Lie group then ω is a left-invariant p -form on G iff ω is a smooth section of $\bigwedge^p M \rightarrow M$ such that $l_g^* \omega = \omega$ for each $g \in G$.

Remark. We will show below that any mapping $\omega: G \rightarrow \bigwedge^p G$ such that $\omega(\bigwedge^p(T_x G)) = x$ for every $x \in G$ is necessarily smooth if it has the property that $l_g^* \omega = \omega$ for all $g \in G$. Recall that the equation $l_g^* \omega = \omega$ means that

$$\omega_x(v_1, \dots, v_p) = (l_g^* \omega)_x(v_1, \dots, v_p) = \omega_{gx}(dl_g(v_1), \dots, dl_g(v_p))$$

for each $x \in G$ and v_1, v_2, \dots, v_p in $T_x G$

Observation 1 If ω is a left-invariant one-form on a Lie group G and X is a left-invariant vector field on G then the mapping defined by $x \mapsto \omega_x(X(x))$ is constant.

$$\begin{aligned}\text{Proof } \omega(\bar{x})(y) &= \omega_y(\bar{x}(y)) = \omega_e(d_{\bar{y}}^{-1}(\bar{x}_y)) \\ &= \omega_e(d_{\bar{y}}^{-1}(d_{\bar{e}}(x))) = \omega_e(x).\end{aligned}$$

Observation 2 If $\omega_e \in \Lambda^p(T_e G)$ for some Lie group G then the mapping $\omega: G \rightarrow \Lambda^p G$ defined by $\omega_x(v_1, \dots, v_p) = \omega_e(d_{x^{-1}}^x(v_1), \dots, d_{x^{-1}}^x(v_p))$ is a left-invariant p -form on G .

Proof. It is clear that ω_x is a multilinear mapping from $T_x G \times \dots \times T_x G$ into \mathbb{R} for each $x \in G$. Once we show $l_g^* \omega = \omega$ for each $g \in G$, it will follow from the next theorem that ω is smooth. Thus we have only show ω is left-invariant. For $g, x \in G$ and $v_1, v_2, \dots, v_p \in T_x G$,

$$\begin{aligned}(l_g^* \omega)(v_1, v_2, \dots, v_p) &= \omega_{gx}(d_{gx}^{-1}(v_1), \dots, d_{gx}^{-1}(v_p)) \\ &= \omega_e(d_{gx}^{-1}(d_{gx}^x(v_1)), \dots, d_{gx}^{-1}(d_{gx}^x(v_p))) \\ &= \omega_e(d_x(l_{g^{-1}} \circ l_g)(v_1), \dots, d_x(l_{g^{-1}} \circ l_g)(v_p)) \\ &= \omega_e(d_x^{-1}(v_1), \dots, d_x^{-1}(v_p)) \\ &= \omega_x(v_1, v_2, \dots, v_p).\end{aligned}$$

Thus $l_g^* \omega = \omega$ as asserted above.

~~Theorem & defn.~~

Theorem If G is a Lie group and $\omega: G \rightarrow \Lambda^p G$ is a mapping such that $\omega(\lambda^p(T_x G)) = x$ for all $x \in G$ and $l_g^* \omega = \omega$ for all $g \in G$, then ω is smooth.

Proof. We show that if Y_1, Y_2, \dots, Y_p are smooth vector fields defined on any open subset U of G then the mapping $\omega(Y_1, \dots, Y_p)$ defined by

$$y \mapsto \omega_y(Y_1(y), \dots, Y_p(y))$$

is smooth. To do this we relate the Y_i 's to a specific choice of left-invariant vector fields $\{\bar{X}_j\}$ and use properties of left-invariance to obtain the result. Let v_1, v_2, \dots, v_r denote a basis of $T_e G$ and let $\bar{X}_i = \bar{X}_G^{v_i}$ for $1 \leq i \leq r$. First observe that $\omega(\bar{X}_{i_1}, \dots, \bar{X}_{i_p})$ is constant for each choice of $1 \leq i_j \leq r$. Indeed

$$\begin{aligned} \omega(\bar{X}_{i_1}, \dots, \bar{X}_{i_p})(x) &= \omega_x(d\ell_x(\bar{X}_{i_1}(e)), \dots, d\ell_x(\bar{X}_{i_p}(e))) \\ &= (l_x^* \omega)_e(v_{i_1}, v_{i_2}, \dots, v_{i_p}) \\ &= \omega_e(v_{i_1}, v_{i_2}, \dots, v_{i_p}) \end{aligned}$$

is clearly constant. Since $\{\bar{X}_j(x)\}_{j=1}^r$ is a basis of $T_x G$ for each $x \in U$, we see that

$$Y_i(x) = \sum_{j=1}^r \lambda_{ij}(x) \bar{X}_j(x)$$

for $\lambda_{ij}(x) \in \mathbb{R}$. Since \bar{X}_j is smooth for each j and since Y_i is smooth on U for $1 \leq i \leq p$ it follows that λ_{ij} is a smooth function from U

into \mathbb{R} for $1 \leq i \leq p, 1 \leq j \leq n$ (Prove it!). It follows that

$$\begin{aligned}\omega(Y_1, \dots, Y_n)(x) &= \omega_x \left(\sum_{j_1=1}^n \lambda_1^{j_1}(x) X_{j_1}(x), \dots, \sum_{j_p=1}^n \lambda_p^{j_p}(x) X_{j_p}(x) \right) \\ &= \sum_{j_1=1}^n \dots \sum_{j_p=1}^n [\lambda_1^{j_1}(x) \dots \lambda_p^{j_p}(x)] \omega_x(X_{j_1}(x), \dots, X_{j_p}(x)) \\ &= \sum_{j_1=1}^n \dots \sum_{j_p=1}^n \omega_x(v_{j_1}, \dots, v_{j_p})(\lambda_1^{j_1} \dots \lambda_p^{j_p})(x)\end{aligned}$$

which is clearly smooth on U . The Theorem follows.

Definition Note that the set of all left-invariant p -forms on a Lie group G is a subspace of the vector space $\Gamma(\Lambda^p G)$ of all differential forms on G . This subspace is denoted $\Gamma_{\text{inv}}(\Lambda^p G)$. Let

$$\Gamma_{\text{inv}}(\Lambda G) = \Gamma_{\text{inv}}\left(\bigoplus_p (\Lambda^p G)\right) = \bigoplus_p \Gamma_{\text{inv}}(\Lambda^p G)$$

Elements of this vector space are denoted by

$$\omega = \omega_0 + \omega_1 + \dots + \omega_p$$

where $\omega_i \in \Gamma_{\text{inv}}(\Lambda^i G)$ for $i \geq 1$ and where $\omega_0 \in \mathbb{R}$.

Note that $\Gamma(\Lambda^0 G)$ is simply $C^\infty(G)$ but the only invariant functions on G are constants since $f(x) = f(l_x(e)) = (f \circ l_x)(e) = f(e)$ for $x \in G$ and f an invariant function on G . We define an operation \wedge on $\Gamma_{\text{inv}}(\Lambda G)$

by requiring that $(\omega_p \wedge \omega_q)(x) = \omega_p(x) \wedge \omega_q(x)$ for

$\omega_p \in \Gamma_{\text{inv}}(\Lambda^p G)$, $\omega_q \in \Gamma_{\text{inv}}(\Lambda^q G)$, and $x \in G$. Thus

if $\omega = \omega_0 + \omega_1 + \dots + \omega_p$, $\tau = \tau_0 + \tau_1 + \dots + \tau_q$

$$\omega \wedge \tau = \left(\sum_{i=0}^p \omega_i \right) \wedge \left(\sum_{j=0}^q \tau_j \right) \equiv \sum_{i,j} (\omega_i \wedge \tau_j)$$

Clearly $(\Gamma_{\text{inv}}(\Lambda G), +, \cdot, \wedge)$ is an associative algebra.

Theorem The algebra $\Gamma_{\text{inv}}(\Lambda G)$ of left-invariant forms on a Lie group G is isomorphic to the exterior algebra $\Lambda(T_e^*G)$ of the vector space T_e^*G .

Proof First let $\psi_p : \Gamma_{\text{inv}}(\Lambda^p G) \rightarrow \Lambda^p(T_e^*G)$ be defined by $\psi_p(\omega) = \omega_e$. Clearly ψ_p is linear. We show that ψ_p is injective. Assume that $\psi_p(\omega) = \omega_e = 0$

if $x \in G$ and $v_1, v_2, \dots, v_p \in T_x G$, then

$$\omega_x(v_1, v_2, \dots, v_p) = \omega_e(d\ell_{x^{-1}}(v_1), \dots, d\ell_{x^{-1}}(v_p)) = 0$$

and so $\text{Ker } \psi_p = \{0\}$. Thus ψ_p is injective.

To see that ψ_p is surjective let $\omega_e \in \Lambda^p(T_e^*G)$ be arbitrary. Define τ by $\tau_x(v_1, v_2, \dots, v_p) = \omega_e(d\ell_{x^{-1}}(v_1), \dots, d\ell_{x^{-1}}(v_p))$ for $x \in G$, $v_1, v_2, \dots, v_p \in T_x G$.

Since

$$\begin{aligned} \tau_{gx}(d\lg(v_1), \dots, d\lg(v_p)) &= \omega_e(d\ell_{gx^{-1}}(d\lg(v_1)), \dots, d\ell_{gx^{-1}}(d\lg(v_p))) \\ &= \omega_e(d\ell_{x^{-1}}(d\lg(gx^{-1} \circ \lg)(v_1)), \dots, d\ell_{x^{-1}}(d\lg(gx^{-1} \circ \lg)(v_p))) \\ &= \omega_e(d\ell_{x^{-1}}(v_1), \dots, d\ell_{x^{-1}}(v_p)) \\ &= \tau_x(v_1, \dots, v_p) \end{aligned}$$

for all $x \in G$, $v_1, \dots, v_p \in T_x G$ we see that $\tau \in \Gamma_{\text{inv}}(\Lambda^p G)$ and $\psi_p(\tau) = \tau_e = \omega_e$. Thus ψ_p is surjective and ψ_p is a vector space isomorphism. Now define

$\tilde{\psi} : \Gamma_{\text{inv}}(\Lambda G) \rightarrow \Lambda(T_e^*G)$ by

$$\tilde{\psi}(\omega_0 + \omega_1 + \dots + \omega_p) = \psi_0(\omega_0) + \psi_1(\omega_1) + \dots + \psi_p(\omega_p).$$

It is straightforward to show that $\tilde{\psi}$ is an algebra isomorphism. The details are left to the reader.

It is obvious that $\widehat{\Psi}$ is linear since Ψ_p is linear for each p . Similarly $\widehat{\Psi}$ is bijective since Ψ_p is bijective for each p . We show it is an algebra homomorphism. Recall that if $\omega_p \in \Gamma(\Lambda^q G)$ and $\omega_q \in \Gamma(\Lambda^r G)$ then $\omega_p \wedge \omega_q$ is in $\Gamma(\Lambda^{p+q} G)$ and is defined by $(\omega_p \wedge \omega_q)(x) = \omega_p(x) \wedge \omega_q(x) = (\omega_p)_x \wedge (\omega_q)_x$ for all $x \in G$. Thus if $\omega = \sum_{p=0}^q \omega_p$ and $\tau = \sum_{l=0}^q \tau_l$ are in $\Gamma_{\text{inv}}(\Lambda G)$, then

$$\begin{aligned}\widehat{\Psi}(\omega \wedge \tau) &= \widehat{\Psi}\left(\sum_{k,l} (\omega_k \wedge \tau_l)\right) = \\ &= \sum_{k,l} \widehat{\Psi}(\omega_k \wedge \tau_l) = \sum_{k,l} (\omega_k \wedge \tau_l)(e) \\ &= \sum_{k,l} [\omega_k(e) \wedge \tau_l(e)] = [\sum_k \omega_k(e)] \wedge [\sum_l \omega_l(e)] \\ &= \widehat{\Psi}(\omega) \wedge \widehat{\Psi}(\tau).\end{aligned}$$

The Theorem follows.

Theorem The vector space $\Gamma_{\text{inv}}(\Lambda^1 G)$ of left-invariant 1-forms of a Lie group G is isomorphic to the vector space dual of the Lie algebra $\mathfrak{g} = \mathfrak{l}(G)$ of G .

Proof. For each $\omega \in \Gamma_{\text{inv}}(\Lambda^1 G)$ let $\widehat{\omega}$ denote the mapping from \mathfrak{g} into \mathbb{R} defined by letting $\widehat{\omega}(X)$ be the constant value of the function $\omega(X)$ for $X \in \mathfrak{g}$ (recall observation 1 above). Now define $\Psi: \Gamma_{\text{inv}}(\Lambda^1 G) \rightarrow \mathfrak{g}^*$ by $\Psi(\omega) = \widehat{\omega}$.

It is obvious that $\hat{\omega}$ is in fact in \mathcal{F}^* and that ψ is linear. To see that ψ is injective assume that $\omega \in \Gamma_{\text{inv}}(\Lambda^1 G)$ such that $\psi(\omega) = 0$. Then $\hat{\omega}(X) = 0$ for each $X \in \mathcal{F}$ and thus $\omega_e(X) = 0$. This implies that $\omega_e(v) = 0$ for every $v \in T_e G$ since we may choose $X = X^v$. Thus $\omega_e = 0$. But $\omega_x = l_{x^{-1}}^* \omega_e$ and so for each $v \in T_x G$, $\omega_x(v) = \omega_e(dl_{x^{-1}}(v)) = 0$. Thus $\omega = 0$ and ψ is injective. It is clear from the last theorem that the mapping $\psi_p : \Gamma_{\text{inv}}(\Lambda^1 G) \rightarrow \Lambda^p(T_e^* G)$ defined by $\psi_p(\omega) = \omega_e$ is an isomorphism for each p .

Thus

$$\begin{aligned}\dim \Gamma_{\text{inv}}(\Lambda^1 G) &= \dim \Lambda^1(T_e^* G) \\ &= \dim(T_e^* G) \\ &= \dim \mathcal{F} = \dim \mathcal{F}^*\end{aligned}$$

Since $\psi : \Gamma_{\text{inv}}(\Lambda^1 G) \rightarrow \mathcal{F}^*$ is an injective linear transformation and the dimensions of its domain and range are the same, it follows that ψ is surjective and so is an isomorphism.

Example. Let $G = \text{GL}(n)$ and recall that the functions $x_{ij} : G \rightarrow \mathbb{R}$ defined by $x_{ij}(A) = A_{ij}^1$ are the components of a chart on G . For each $1 \leq i, j \leq n$ let

$$\omega_{ij} = \sum_{k=1}^n x_{ik}^{-1} dx_{kj}.$$

Then $\{\omega_{ij} \mid 1 \leq i, j \leq n\}$ is a basis of left-invariant

1 -forms on G . To see that this is true we show that w_{ij} is left-invariant for each i, j . Since the forms are clearly independent and since $\dim(\Gamma_{\text{inv}}(\Lambda^1 G)) = n^2$, they obviously form a basis of $\Gamma_{\text{inv}}(\Lambda^1 G)$.

Let $A \in G$, then

$$l_A^* w_{ij} = l_A^* \left(\sum_k x_{ik}^{-1} dx_{kj} \right) = \sum_k (x_{ik}^{-1} \circ l_A) d(x_{kj} \circ l_A).$$

But

$$\begin{aligned} (x_{kj} \circ l_A)(B) &= x_{kj}(AB) = (AB)_{kj} = \sum_l A_{kl} B_{lj} \\ &= \sum_l A_{kl} x_{lj}(B) = \left(\sum_l A_{kl} x_{lj} \right)(B) \end{aligned}$$

for each $B \in G$. Thus $x_{kj} \circ l_A = \sum_l A_{kl} x_{lj}$.

Similarly

$$\begin{aligned} (x_{ik}^{-1} \circ l_A)(B) &= x_{ik}^{-1}(AB) = (AB)_{ik}^{-1} \\ &= \sum_l B_{il}^{-1} A_{lk}^{-1} = \left(\sum_l A_{lk}^{-1} x_{il}^{-1} \right)(B) \end{aligned}$$

for $B \in G$. So

$$\begin{aligned} \sum_k (x_{ik}^{-1} \circ l_A) d(x_{kj} \circ l_A) &= \sum_k \left(\sum_l A_{lk}^{-1} x_{il}^{-1} \right) d \left(\sum_p A_{kp} x_{pj} \right) \\ &= \sum_{k,l,p} A_{lk}^{-1} x_{il}^{-1} A_{kp} dx_{pj} \\ &= \sum_{l,p} \left(\sum_k A_{lk}^{-1} A_{kp} \right) x_{il}^{-1} dx_{pj} \\ &= \sum_l x_{il}^{-1} dx_{lj} = w_{ij}. \end{aligned}$$

It follows that w_{ij} is left-invariant and by a previous theorem it is necessarily smooth.

There is a differential form Θ called the Maurer-Cartan form which is indispensable to the development of a theory of curvature induced by connections on a Principal fiber bundle. This form actually characterizes so-called flat connections, those which have zero curvature. This form is also identified with the Faddeev-Popov ghost which occurs in non-abelian gauge theory.

Definition Let G be a Lie group. The Maurer-Cartan form of G is a $\mathfrak{g} = \mathfrak{l}(G)$ -valued 1-form on G subject to the conditions:

- (1) Θ is left-invariant, i.e. $l_g^* \Theta = \Theta$ for all $g \in G$
- (2) for each left-invariant vector field X on G $\Theta(X) = X$, i.e., for $x \in G$ $\Theta_x(X_x) = \overline{X}$.

Remark. It is not immediately clear that such a form exists or that it is unique. To show that such a form exists choose a basis v_1, v_2, \dots, v_m of $T_x G$. For $w \in T_x G$, $x \in G$, write

$$w = \sum \lambda^i X^{v_i}(x)$$

for $\lambda^1, \lambda^2, \dots, \lambda^m \in \mathbb{R}^i$. Define

$$\Theta_x(w) = \sum_i \lambda^i X^{v_i} = \overline{\sum_i \lambda^i v_i}$$

This defines a 1-form with values in \mathfrak{g} , but we should show that it is independent of the choice of basis $\{v_i\}$ and that it is left invariant. satisfies conditions (1) and (2)

Notice that if $\{V_i\}$ is another basis of $T_e G$, then $V_i = \sum_j A_{ij} V_j$ for $A_{ij} \in \mathbb{R}$. Thus $w = \sum_i \lambda^i \sum V_i(x)$

for $\lambda^1, \lambda^2, \dots, \lambda^n \in \mathbb{R}$ and

$$\sum_j \lambda^j \sum V_j(x) = w = \sum_i \lambda^i \sum_j A_{ij} V_j(x) = \sum_{i,j} \lambda^i A_{ij} V_j(x).$$

It follows that $\lambda^j = \sum_i \bar{A}_{ij} \lambda^i$ for each $1 \leq j \leq n$

and

$$\textcircled{H}_x(w) = \sum_j \lambda^j V_j = \sum_{i,j} \bar{A}_{ij} \lambda^i V_j = \sum_i \lambda^i V_i.$$

It follows that the definition of \textcircled{H} does not depend on which basis of $T_e G$ is used to define it.

We must now show that the two conditions (1) and (2) are satisfied. Let $v_1, v_2, \dots, v_n \in T_e G$ be a basis

of $T_e G$. If \bar{X} is any left-invariant vector field on G and $x \in G$, then $\bar{X}(x) = \sum_i \lambda^i \sum V_i(x)$ for $\lambda^i \in \mathbb{R}$ and

$$\textcircled{H}_x(\bar{X}(x)) = \sum_i \lambda^i \sum V_i.$$

Since \bar{X} and $\sum_i \lambda^i \sum V_i$ are both left-invariant and since they agree at one point $x \in G$, they agree at every point (Prove this!). Thus

$$\textcircled{H}_x(\bar{X}(x)) = \sum_i \lambda^i \sum V_i = \bar{X}.$$

So condition (1) follows. We now show that \textcircled{H} is left-invariant. Let $x, g \in G$ we show that

$$(\textcircled{H}_x)^*(\textcircled{H})_g = (\textcircled{H}_g)^*(\textcircled{H}_x). \text{ Let } w \in T_x G \text{ and write } w = \sum_i \lambda^i \sum V_i(x),$$

as above. Then

$$(\textcircled{H}_g)^*(\textcircled{H})_x(w) = \sum_i \lambda^i \textcircled{H}_{gx} \left(d\lg \left(\sum V_i(x) \right) \right) = \sum_i \lambda^i \textcircled{H}_{gx} \left(\sum V_i(gx) \right)$$

$$= \Theta_{gx} \left(\sum_i \lambda^i X^{v_i}(gx) \right) = \sum_i \lambda^i X^{v_i} = \Theta_x(w).$$

So $\Theta_x(w) = (\lg^* \Theta)_x(w)$ for all $w \in T_x G$ and all $x \in G$;
consequently $\lg^* \Theta = \Theta$ for all $g \in G$.

To see that there exists a unique such Θ
assume Θ_1 and Θ_2 are both \mathfrak{g} -valued 1-forms on G
satisfying (1) and (2). We first show that $(\Theta_1)_e = (\Theta_2)_e$.
Let $v \in T_e G$, then $\Theta_1(X^v) = X^v$ and $\Theta_2(X^v) = X^v$
(by (1)) and

$$(\Theta_1)_e(v) = (\Theta_1)_e(X^v(e)) = X^v = (\Theta_2)_e(X^v(e)) = (\Theta_2)_e(v).$$

Since v was arbitrary, $(\Theta_1)_e = (\Theta_2)_e$. By
left-invariance

$$\begin{aligned} (\Theta_1)_x &= (\lg_x^* \Theta_1)_x = (\Theta_1)_e \circ d\lg_x^{-1} \\ &= (\Theta_2)_e \circ d\lg_x^{-1} = (\Theta_2)_x \end{aligned}$$

for all $x \in G$. So $\Theta_1 = \Theta_2$ and the form Θ is
uniquely defined by (1) and (2).

Definition Let G be a Lie group and ω a \mathfrak{g} -valued
differential 1-form on G . If $\{X_a\}$ is a basis
of the Lie algebra \mathfrak{g} and $\{\varphi^a\}$ is the basis of \mathfrak{g}^*
dual to \mathfrak{g} then the real-valued 1-forms
 $\omega^a = \varphi^a \circ \omega$ are called the components of ω
relative to the basis $\{X_a\}$. Moreover we will
say that the components $\{\omega^a\}$ of ω are dual
to $\{X_a\}$ iff for each $x \in G$, $\{\omega_x^a\}$ is the
basis of $T_x^* G$ dual to the basis $\{X_a(x)\}$ of $T_x G$.
In this case $\omega_x^a(X_b(x)) = \delta_a^b$ for all $x \in G$.
Note that in general the components $\{\omega^a\}$ of a

form ω may be arbitrary 1-forms since for every set of 1-forms $\{\omega^a\}$, $\omega = \omega^a \bar{x}_a$ is a \mathfrak{g} -valued 1-form with components $\{\omega^a\}$.

Theorem Let G be a Lie group and ω a \mathfrak{g} -valued 1-form on G , then

- (1) ω is left-invariant iff its components relative to some basis of \mathfrak{g} are left-invariant, and
- (2) ω is the Maurer-Cartan form Θ iff it is left-invariant and its components relative to some basis $\{\bar{x}_a\}$ of \mathfrak{g} are dual to $\{\bar{x}_a\}$.

Proof Let $\{\bar{x}_a\}$ denote a basis of $\mathfrak{g} = l(G)$ and $\{\omega^a\}$ the components of ω relative to $\{\bar{x}_a\}$. We show that ω is left-invariant iff ω^a is left-invariant for each a . For $g \in G$,

$$\begin{aligned} l_g^* \omega &= \omega \iff l_g^*(\omega^a \bar{x}_a) = \omega^a \bar{x}_a \\ &\iff (l_g^* \omega^a) \bar{x}_a = \omega^a \bar{x}_a \\ &\iff l_g^* \omega^a = \omega^a \quad \text{for all } a. \end{aligned}$$

Thus (1) holds.

Consider (2). First note that if $\omega = \Theta$ is the Maurer-Cartan form then ω is left-invariant and $\omega(\bar{x}) = \bar{x}$ for each left-invariant vector field \bar{x} . Thus $\omega(\bar{x}_b) = \bar{x}_b$ for each b and

$\oint_b \bar{x}_a = \bar{x}_b = \omega_x(\bar{x}_b) = \omega_x^a(\bar{x}_b(x)) \bar{x}_a$. It follows that $\omega_x^a(\bar{x}_b(x)) = \delta_{ab}^a$ and so $\{\omega_x^a\}$ is the basis of T_x^*G dual to $\{\bar{x}_a(x)\}$. Conversely, assume $\omega = \omega^a \bar{x}_a$ is

left-invariant and that the components $\{\omega^a\}$ of ω relative to a basis $\{X_a\}$ of \mathfrak{g} are dual to $\{X_a\}$. Since $\{\omega^a\}$ are dual to $\{X_a\}$ we have that $\omega_x^a(X_b(x)) = \delta_b^a$ for all $x \in G$ and for all a, b .

Thus

$$\omega_x(X_b(x)) = \omega_x^a(X_b(x)) X_a = \delta_b^a X_a = X_b = \Theta_x(X_b(x))$$

for all b . Since $\{X_b(x)\}$ is a basis of $T_x G$ and $\omega_x, \Theta_x \in T_x^* G$ it follows that $\omega_x = \Theta_x$ for each $x \in G$. So $\Theta = \omega$ and the theorem follows.

An important equation in the subsequent development is the Maurer-Cartan equation. This equation is also called the structure equation and is

$$d\Theta + [\Theta, \Theta] = 0$$

Here $[\Theta, \Theta]$ is the \mathfrak{g} -valued 2-form defined on G by

$$[\Theta, \Theta]_x(v, w) = [\Theta_x(v), \Theta_x(w)]$$

for $x \in G$ and $v, w \in T_x G$. It turns out that this equation is most easily derived in terms of the components $\{\Theta^a\}$ of Θ relative to an arbitrary basis. In terms of components the structure equation takes the form

$$d\Theta^a + f_{bc}^a(\Theta^b \wedge \Theta^c) = 0$$

where $\{f_{bc}^a\}$ are the so-called structure constants of the Lie algebra \mathfrak{g} . These constants are defined as follows.

Definition Let \mathfrak{g} be any finite-dimensional Lie algebra and let $\{\mathbb{X}_a\}$ be a basis of \mathfrak{g} . The structure constants of \mathfrak{g} relative to the basis $\{\mathbb{X}_a\}$ is the set of numbers $\{f_{bc}^a\}$ such that

$$[\mathbb{X}_b, \mathbb{X}_c] = f_{bc}^a \mathbb{X}_a.$$

Remark Given a finite-dimensional Lie algebra \mathfrak{g} and a choice $\{\mathbb{X}_a\}$ of a basis of \mathfrak{g} the structure constants $\{f_{bc}^a\}$ satisfy the identities

$$(1) \quad f_{bc}^a = -f_{ca}^b \quad \text{for all } a, b, c$$

$$(2) \quad f_{bc}^p f_{ap}^q + f_{ca}^p f_{bp}^q + f_{ab}^p f_{cp}^q = 0, \quad \text{for all } a, b, c, q.$$

These identities are an immediate consequence of the skew-symmetry of the Lie-bracket and the Jacobi identity. Conversely, if one is given a set of constants $\{f_{bc}^a\}$ which satisfies (1) and (2) and if $\{\mathbb{X}_a\}$ is a basis of a vector space \mathfrak{g} then one can define a bracket $[\cdot, \cdot]$ on \mathfrak{g} by

$$[\mathbb{X}_b, \mathbb{X}_c] = f_{bc}^a \mathbb{X}_a$$

and \mathfrak{g} will be a Lie algebra with structure constants $\{f_{bc}^a\}$ relative to $\{\mathbb{X}_a\}$.

To prove the component form of the Maurer-Cartan equation we first need a Lemma.

Lemma If ω is a real-valued one-form on a manifold M then $d\omega(\mathbb{X}, \mathbb{Y}) = \mathbb{X}(\omega(\mathbb{Y})) - \mathbb{Y}(\omega(\mathbb{X})) - \omega([\mathbb{X}, \mathbb{Y}])$ for each pair of vector fields \mathbb{X}, \mathbb{Y} on M .

Proof It suffices to prove the identity locally in terms of an admissible chart (U, x) . Write

$$\underline{X} = \underline{X}^\mu \frac{\partial}{\partial x^\mu}$$

$$\underline{Y} = \underline{Y}^\nu \frac{\partial}{\partial x^\nu}$$

$$\omega = \omega_\gamma dx^\gamma.$$

Then observe that

$$[\underline{X}, \underline{Y}] = [\underline{X}(\underline{Y}^\eta) - \underline{Y}(\underline{X}^\eta)] \frac{\partial}{\partial x^\eta}$$

and

$$\underline{X}(\omega(\underline{Y})) - \underline{Y}(\omega(\underline{X})) - \omega([\underline{X}, \underline{Y}])$$

$$= \underline{X}(\omega_\gamma \underline{Y}^\nu) - \underline{Y}(\omega_\mu \underline{X}^\mu) - \omega_\eta (\underline{X}(\underline{Y}^\eta) - \underline{Y}(\underline{X}^\eta))$$

$$= \underline{X}^\mu \partial_\mu (\omega_\gamma \underline{Y}^\nu) - \underline{Y}^\nu \partial_\nu (\omega_\mu \underline{X}^\mu)$$

$$- \omega_\nu (\underline{X}^\mu \partial_\mu \underline{Y}^\nu) + \omega_\mu (\underline{Y}^\nu \partial_\nu \underline{X}^\mu)$$

$$= (\partial_\mu \omega_\nu)(\underline{X}^\mu \underline{Y}^\nu) - (\partial_\nu \omega_\mu)(\underline{X}^\mu \underline{Y}^\nu)$$

$$= (\partial_\mu \omega_\nu)(dx^\mu \wedge dx^\nu)(\underline{X}, \underline{Y})$$

$$= d\omega(\underline{X}, \underline{Y}).$$

Corollary If $\{\Theta^a\}$ are the components of the Maurer-Cartan form relative to some basis of left-invariant vector fields $\{\underline{X}_a\}$ on a Lie group G , then

$$d\Theta^a + \frac{1}{2} f_{bc}^a (\Theta^b \wedge \Theta^c) = 0$$

for all a . Here $\{f_{bc}^a\}$ denotes the structure constants of $\mathfrak{g} = \mathfrak{l}(G)$ relative to the basis $\{\underline{X}_a\}$.

Proof. By the lemma we have

$$d\Theta^a(\bar{X}_p, \bar{X}_q) = \bar{X}_p(\Theta^a(\bar{X}_q)) - \bar{X}_q(\Theta^a(\bar{X}_p)) - \Theta^a([\bar{X}_p, \bar{X}_q])$$

for each a . Recall that $\Theta^a(\bar{X}_c)$ is constant for each a, c and so $\bar{X}_p(\Theta^a(\bar{X}_q)) = 0$, $\bar{X}_q(\Theta^a(\bar{X}_p)) = 0$.

Thus

$$d\Theta^a(\bar{X}_p, \bar{X}_q) = -\Theta^a(f_{pq}^c \bar{X}_c) = -f_{pq}^c \delta_c^a = -f_{pq}^a.$$

But

$$\begin{aligned} \cancel{f_{bc}^a (\Theta^b \wedge \Theta^c)(\bar{X}_p, \bar{X}_q)} &= f_{bc}^a [\Theta^b(\bar{X}_p) \Theta^c(\bar{X}_q) - \Theta^b(\bar{X}_q) \Theta^c(\bar{X}_p)] \\ &= f_{bc}^a [\delta_p^b \delta_q^c - \delta_q^b \delta_p^c] \\ &= f_{pq}^a - f_{qp}^a = 2f_{pq}^a \end{aligned}$$

and so

$$d\Theta^a(\bar{X}_p, \bar{X}_q) = -\frac{1}{2} f_{bc}^a (\Theta^b \wedge \Theta^c)(\bar{X}_p, \bar{X}_q)$$

for all a, p, q . Thus

$$d\Theta^a = -\frac{1}{2} f_{bc}^a (\Theta^b \wedge \Theta^c).$$

We now derive the Maurer-Cartan equation in coordinate-free form:

$$d\Theta + \frac{1}{2} [\Theta, \Theta] = 0.$$

To see this we first expand $[\Theta, \Theta]$. For p, q arbitrary

$$[\Theta, \Theta](\bar{X}_p, \bar{X}_q) = [\Theta(\bar{X}_p), \Theta(\bar{X}_q)] = \Theta^a(\bar{X}_p) \Theta^b(\bar{X}_q) [\bar{X}_a, \bar{X}_b]$$

$$= \delta_p^a \delta_q^b f_{ab}^c X_c$$

$$= f_{pq}^c X_c = \frac{1}{2} f_{ab}^c (\Theta^a \wedge \Theta^b)(X_p, X_q) X_c$$

$$= - d\Theta^c(X_p, X_q) X_c$$

$$= - d\Theta(X_p, X_q).$$

The structure equation follows.