

Definition Let M be a manifold and G a Lie group. Clearly the projection $\pi_M: M \times G \rightarrow M$ is a trivial fiber bundle with fiber G . Define a right action of G on $M \times G$ by $(x, a) \cdot g = (x, ag)$, $x \in M$, $a, g \in G$. Notice that this action is smooth and that the orbits are precisely the fibers of π_M . Any fiber bundle obtained in this way is called a trivial principal G -bundle. More generally assume $\pi: P \rightarrow M$ is any fiber bundle with fiber G . We say that it is a principal G -bundle or simply a principal fiber bundle iff

- (1) there is a free right action of G on P
- (2) $\pi: P \rightarrow M$ is locally G -bundle isomorphic to a trivial principal G -bundle in the sense that there is a local trivialization $\{\psi_\alpha\}_{\alpha \in U}$ of $\pi: P \rightarrow M$

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi_U} & U \times G \\ \pi \searrow & & \downarrow \pi_G \\ & U & \end{array}$$

such that $\psi_{\alpha}(x \cdot g) = \psi_{\alpha}(x) \cdot g$ for $x \in \pi^{-1}(U)$, $g \in G$.

Example 1 The frame bundle $\pi: \mathcal{F}M \rightarrow M$ defined by $\pi(x, \{e_i\}) = x$ where $x \in M$ and $\{e_i\}$ is a basis of $T_x M$ has already been shown to be a fiber bundle. Define an action of $\mathbb{GL}(n)$ on $\mathcal{F}M$ by

$$(x, \{e_i\}) \cdot g = (x, \{e_i g_j^T\}), \quad g \in \mathbb{GL}(n)$$

Then this action is a free right action and it becomes

Example 2 Let M be a manifold and g a metric on M . Let $\Omega M = \{(x, \{e_i\}) \in FM \mid \{e_i\}$ is a g -orthonormal basis $\}$. Let $\tau: \Omega M \rightarrow M$ be defined by $\tau(x, \{e_i\}) = x$. If g is a type (p, q) metric then define an action of $O(p, q)$ on ΩM by $(x, \{e_i\}) \cdot g = (x, \{e_j g_{ij}^k\})$, $g \in O(p, q)$. This bundle is a principal $O(p, q)$ bundle and is a "sub-bundle" of $FM \xrightarrow{\pi} M$ in a sense to be made precise later in the exposition.

Example 3 Let G be any Lie group and H any closed subgroup. Then $\pi: G \rightarrow G/H$ is a fiber bundle with fiber H . If we define an action $G \times H \rightarrow G$ by $(g, h) \mapsto gh$ then H acts freely on the right of G and $\pi: G \rightarrow G/H$ is a principal H -bundle. In particular $SO(n+1) \rightarrow SO(n+1)/SO(n)$ is a principal $SO(n)$ -bundle with total space $SO(n+1)$ and base space the n -sphere.

Example 4 Let $\pi: E \rightarrow M$ be any vector bundle with fiber a finite dimensional vector space V . Let

$FE = \{(x, \{v_i\}) \mid x \in M \text{ and } \{v_i\} \text{ is a basis of } \pi^{-1}(x)\}$
 if $r = \dim V$ define a right action of $GL(r)$ on FE by $(x, \{v_i\}) \cdot g = (x, \{v_j g_{ji}^k\})$, $g \in GL(r)$. Note that if $(x, \{v_i\}) \cdot g = (x, \{v_i\})$ then $v_j g_{ji}^k = v_j \delta_{ji}^k$ and $g = I$. So the action is a free right action of $GL(r)$ on FE .

Let $\{\psi_\alpha\}_{\alpha \in U}$ be a local trivialization of $\pi: E \rightarrow M$

as a vector bundle, thus $\psi_\alpha|_{\pi^{-1}(x)}: \pi^{-1}(x) \rightarrow \{x\} \times V$ is linear for each x . Recall that $\psi_\alpha|_{\pi^{-1}(x)}$ is actually a

vector space isomorphism and so a basis $\{V_i\}$ of $\pi^{-1}(x)$ is transformed to a basis $\{\pi_{V_j}(\psi_0(V_i))\}$ of V (here π_V is the projection of $U \times V$ onto V). Consequently, if we define $JU = \{(x, \{V_i\}) \mid x \in U \text{ and } \{V_i\} \text{ is a basis of } \pi^{-1}(x)\}$ then ψ_0 induces a mapping from JU onto $U \times JV$ where JV is the set of all bases of V . This induced mapping is not quite adequate as we want to obtain a mapping from JU onto $U \times \mathcal{L}(r)$ and to use this mapping to obtain a local trivialization of JE .

So fix a specific basis $\{V_i^0\}$ of V and define a mapping $g : JV \rightarrow \mathcal{L}(r)$ by $g(\{V_i\}) = (V_0^i(V_j))$ where $\{V_0^i\}$ is the basis of V^* (dual of V) dual to $\{V_i\}$. It is easy to see that g is a bijection and so JV may be identified with $\mathcal{L}(r)$ via g .

Finally we define $\tilde{\psi}_0 : JU \rightarrow U \times \mathcal{L}(r)$ by $\tilde{\psi}_0(x, \{V_i\}) = (x, g(\pi_V(\psi_0(V_i))))$. We give JV a differentiable structure by requiring that g be a diffeomorphism and we give JU a differentiable structure by requiring that $\tilde{\psi}_0$ be a diffeomorphism.

Assume that $U_1, U_2 \subset U$ such that $U_1 \cap U_2 \neq \emptyset$. We must show that $\tilde{\psi}_{U_2} \circ \tilde{\psi}_{U_1}^{-1} : (U_1 \cap U_2) \times \mathcal{L}(r) \rightarrow (U_1 \cap U_2) \times \mathcal{L}(r)$ is a diffeomorphism. First observe that $g^{-1} : \mathcal{L}(r) \rightarrow JV$ is given by $g^{-1}(a) = \{a^j_i V_j^0\}$. Next note that

since $\{a^j_i V_j^0\}$ is a basis of V , $\{\psi_{U_1}^{-1}(x, a^j_i V_j^0)\}$ is a basis of $\pi^{-1}(x)$ for $x \in U_1$. Thus

$$\tilde{\psi}_{U_1}^{-1} : U_1 \times \mathcal{L}(r) \rightarrow JU_1$$

is given by $\tilde{\psi}_{U_1}^{-1}(x, a) = (x, \{\psi_{U_1}^{-1}(x, a^j_i V_j^0)\})$

it now follows that

which is a composite of smooth mappings and so is itself smooth. Note that

$$\begin{aligned}
 g(\{v_i\} \cdot h) &= g(\{v_j h^j\}) \\
 &= v_0^k (v_j h^j)_k \\
 &= v_0^k (v_j h^j)_k \\
 &= v_0^k (v_j) h^j_k \\
 &= (v_0^k (v_j)) \cdot h^j_k \\
 &= g(\{v_i\}) \cdot h
 \end{aligned}$$

for $\{v_i\} \in FT$, $h \in \mathcal{D}(n)$. It follows that

$$\begin{aligned}
 \widetilde{\psi}_0((x, \{v_i\}) \cdot h) &= \widetilde{\psi}_0(x, \{v_j h^j\}) \\
 &= (x, g(\{\pi_V(\psi_0(v_j h^j))\})) \\
 &= (x, g(\{\pi_V(\psi_0(v_j)) h^j\})) \\
 &= (x, g(\{\pi_V(\psi_0(v_j))\}) h) \\
 &= \widetilde{\psi}_0(x, \{v_i\}) \cdot h
 \end{aligned}$$

for $(x, \{v_i\}) \in FU$, $h \in \mathcal{D}(n)$. It follows that $FE \xrightarrow{\pi} M$ is a principal $\mathcal{D}(n)$ -bundle.

The primary feature of any principal G -bundle $\pi: P \rightarrow M$ is that it is locally isomorphic to a trivial G -bundle. In a sense such a bundle consists of a family of trivial G -bundles glued together in a nontrivial manner. The idea can be made more precise as follows.

Let $\{\psi_{U_\alpha}\}_{\alpha \in \mathcal{A}}$ be a local trivialization of $\pi: P \rightarrow M$. Write $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ and consider $\alpha, \beta \in \mathcal{A}$ such that $U_\alpha \cap U_\beta \neq \emptyset$. Since $\psi_{U_\alpha}: \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$

has the property that $\psi_{U_\alpha}(u \cdot g) = \psi_{U_\alpha}(u) \cdot g$ for $u \in U_\alpha$ and $g \in G$. we have

$$(\psi_{U_\beta} \circ \psi_{U_\alpha}^{-1})(x, a) = \psi_{U_\beta}(\psi_{U_\alpha}^{-1}(x, a) \cdot a) = (\psi_{U_\beta} \circ \psi_{U_\alpha}^{-1})(x, a)$$

for $x \in U_\alpha \cap U_\beta$ and $a \in G$. Define mappings $g_{\beta\alpha}: U_\alpha \cap U_\beta \rightarrow$ by requiring that $g_{\beta\alpha}(x)$ be the projection of the element $(\psi_{U_\beta} \circ \psi_{U_\alpha}^{-1})(x, a)$ onto G (recall that $(\psi_{U_\beta} \circ \psi_{U_\alpha}^{-1})$ maps $(U_\alpha \cap U_\beta) \times G$ onto $(U_\alpha \cap U_\beta) \times G$). Thus

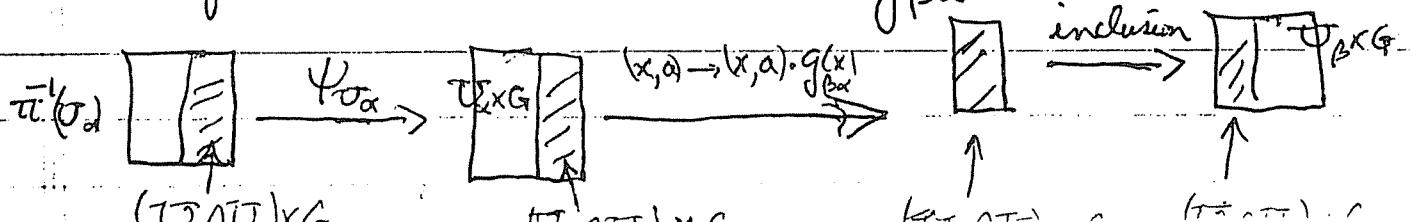
$$(\psi_{U_\beta} \circ \psi_{U_\alpha}^{-1})(x, a) = (x, g_{\beta\alpha}(x)) \cdot a = (x, g_{\beta\alpha}(x) \cdot a)$$

for all $x \in U_\alpha \cap U_\beta$ and $a \in G$.

Intuitively one has that $\pi^{-1}(U_\alpha)$ is diffeomorphic to $U_\alpha \times G$ and $\pi^{-1}(U_\beta)$ is diffeomorphic to $U_\beta \times G$. On the overlap $\pi^{-1}(U_\alpha \cap U_\beta)$ one has two ways of identifying $\pi^{-1}(U_\alpha \cap U_\beta)$ with points of $(U_\alpha \cap U_\beta) \times G$. Due to the fact that the identifications $\psi_{U_\alpha}, \psi_{U_\beta}$ commute with the group action one can translate ψ_{U_α} along its fibers via $g_{\beta\alpha}$ to get ψ_{U_β} , i.e.,

$$\psi_{U_\beta} = (\psi_{U_\beta} \circ \psi_{U_\alpha}^{-1}) \circ \psi_{U_\alpha}$$

where $\psi_{U_\beta} \circ \psi_{U_\alpha}^{-1}$ translates $(x, a) \in (U_\alpha \cap U_\beta) \times G$ along the fiber over x to $(x, a) \cdot g_{\beta\alpha}(x)$.



We see that each local trivialization $\{\Psi_{U_\alpha}\}_{\alpha \in \Lambda}$ defines mappings $g_{\beta\alpha}: U_\alpha \cap U_\beta \rightarrow G$. Moreover if one works on $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ then

$$\Psi_{U_\gamma} \circ \Psi_{U_\alpha}^{-1} = (\Psi_{U_\gamma} \circ \Psi_{U_\beta}^{-1}) \circ (\Psi_{U_\beta} \circ \Psi_{U_\alpha}^{-1})$$

and so evaluating at (x, e) we have

$$(x, g_{\gamma\alpha}(x)) = (x, g_{\gamma\beta}(x) g_{\beta\alpha}(x))$$

and

$$g_{\gamma\alpha}(x) = g_{\gamma\beta}(x) g_{\beta\alpha}(x)$$

for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$. Applying this property when $\alpha = \beta = \gamma$ gives $g_{\alpha\alpha}(x) = e$ (the identity of G) for all $x \in U_\alpha$. In case $\alpha = \gamma$ we obtain

$$g_{\beta\alpha}(x) = g_{\alpha\beta}(x)^{-1}$$

for $x \in U_\alpha \cap U_\beta$. The family of mappings $\{g_{\beta\alpha}\}$ is called the family of transition functions of the local trivialization $\{\Psi_{U_\alpha}\}$. The property

$$g_{\gamma\alpha} = g_{\gamma\beta} g_{\beta\alpha}$$

is called a cocycle condition.

Theorem Let M be a manifold, G a Lie group, and $\{U_\alpha\}_{\alpha \in \Lambda}$ an open covering of M such that for each pair $\alpha, \beta \in \Lambda$ for which $U_\alpha \cap U_\beta \neq \emptyset$ one has a mapping $g_{\beta\alpha}: U_\alpha \cap U_\beta \rightarrow G$. If the family of maps $\{g_{\beta\alpha}\}$ satisfies the cocycle condition then there is a principal G -bundle $\pi: P \rightarrow M$ and a local trivialization of π with transition functions $\{g_{\beta\alpha}\}$.

Proof Let S denote the disjoint union of the family of set $\{U_\alpha \times G\}_{\alpha \in \Lambda}$. One way to think of S is as the set of all order triples (α, x, g) where $\alpha \in \Lambda$, $x \in U_\alpha$, $g \in G$. Since $U_\alpha \times G$ is a manifold it is easy to see that S is also a manifold. Indeed, choose an atlas A_α of admissible charts of $U_\alpha \times G$ for each $\alpha \in \Lambda$ and let $A = \bigcup_{\alpha \in \Lambda} A_\alpha$. If $(\alpha, x_\alpha) \in A_\alpha$ and $(\beta, x_\beta) \in A_\beta$ and $\alpha \neq \beta$ then $U_\alpha \cap U_\beta = \emptyset$. Thus A is an atlas of S and for each α , $S_\alpha = \{\alpha\} \times U_\alpha \times G$ is an open submanifold of S . Define a relation \sim on S as follows: $(\alpha, x, g) \sim (\beta, y, h)$ iff $x, y \in U_\alpha \cap U_\beta$, $y = x$ and $g|_{U_\alpha}(x) = h$. Observe that $(\alpha, x, g) \sim (\alpha, y, h)$ iff $y = x$ and $g = h$ since $g(x) = e$. It is easy to show that \sim is an equivalence relation. Denote the equivalence class containing $(\alpha, x, g) \in S$ by $[\alpha, x, g]$ and let $P = S/\sim$ denote the set of all such classes. Define a right action of G on S by $(\alpha, x, g) \cdot a = (\alpha, x, ga)$ for $(\alpha, x, g) \in S$ and $a \in G$. If $(\alpha, x, g) \sim (\beta, y, h)$ then $(\alpha, x, ga) \sim (\beta, y, ha)$ for each $a \in G$ and so we have a well-defined right action of G on P defined by $[\alpha, x, g] \cdot a = [\alpha, x, ga]$. It is easy to show that G acts freely on P .

Define a mapping $\tilde{\pi}: S \rightarrow M$ by $\tilde{\pi}(\alpha, x, g) = x$. Clearly $\tilde{\pi}(S_\alpha) = U_\alpha$ and $\tilde{\pi}|_{S_\alpha}$ is smooth. So $\tilde{\pi}$ is smooth (since S_α is an open submanifold of S for each α). Note that if $(\alpha, x, g) \sim (\beta, y, h)$ then $x, y \in U_\alpha \cap U_\beta$, $y = x$, and $g|_{U_\alpha}(x) = h$ and consequently $\tilde{\pi}(\beta, y, h) = y = x = \tilde{\pi}(\alpha, x, g)$. Thus $\tilde{\pi}$ is constant

on equivalence classes of S and thus induces a well-defined mapping $\pi: P \rightarrow M$ defined by $\pi([\alpha, x, g]) = x$, $(\alpha, x, g) \in S$.

Let $u \in P$, we show that $\pi^{-1}(\pi(u)) = u \cdot G$. Let

$u = [\alpha, x, g]$ and let $w = [\beta, y, h] \in \pi^{-1}(\pi(u))$. Then

$\pi(u) = \pi(w)$ and thus $y = x$. We see then that $U_\alpha \cap U_\beta \neq \emptyset$ as it contains $y = x$. It follows that $(\alpha, x, g) \sim (\beta, y, g(x)g)$ or that $u = [\alpha, x, g] = [\beta, y, g(x)g] = [\beta, y, h](h^{-1}g(x)g)$ $= w \cdot (h^{-1}g(x)g)$. We see that $w = u(g^{-1}g(x)^{-1}h) \in u \cdot G$.

We have shown that $w \in \pi^{-1}(\pi(u)) \Rightarrow w \in u \cdot G$ and

so $\pi^{-1}(\pi(u)) \subset u \cdot G$. Now assume $w \in u \cdot G$,

Then $w = u \cdot h$ for some h and thus

$w = [\alpha, x, gh]$. It follows that $\pi(w) = x = \pi(u)$ and $w \in \pi^{-1}(\pi(u))$. Thus $u \cdot G \subset \pi^{-1}(\pi(u))$ and

$\pi^{-1}(\pi(u)) = u \cdot G$.

Let $\eta: S \rightarrow P$ denote the mapping defined by $\eta(\alpha, x, g) = [\alpha, x, g]$. We have already observed that $(\alpha, x, g) \sim (\alpha, y, h)$ iff $y = x$ and $h = g$; it follows that $\eta|_{S_\alpha}$ is a bijection from S_α onto $\eta(S_\alpha)$. Since $\eta(u \cdot a) = \eta(u) \cdot a$ for $u \in S$, $a \in G$ and since each orbit of G in S which intersects S_α actually is a subset of S_α we see that $\eta(S_\alpha)$ contains each orbit of G in P which intersects $\eta(S_\alpha)$. Now $\pi(\eta(S_\alpha)) = U_\alpha$

and $\pi^{-1}(U_\alpha)$ is the union of all orbits of G in P

which project to points of U_α , thus $\eta(S_\alpha) = \pi^{-1}(U_\alpha)$.

Moreover if $i_\alpha: U_\alpha \times G \rightarrow S_\alpha$ is defined by $i_\alpha(x, g) = (\alpha, x, g)$ and if $\pi_\alpha: U_\alpha \times G \rightarrow U_\alpha$ is the projection of $U_\alpha \times G$ onto U_α then we have a commutative diagram

$$\begin{array}{ccccc}
 U_\alpha \times G & \xrightarrow{i_\alpha} & S_\alpha & \xrightarrow{n|S_\alpha} & \pi^{-1}(U_\alpha) \\
 \downarrow \pi_\alpha & & \downarrow \hat{\pi}|S_\alpha & & \downarrow \pi \\
 U_\alpha & & & &
 \end{array}$$

We will show that P possesses a differentiable structure such that the maps ψ_{U_α} defined by $\psi_{U_\alpha} = [(n|S_\alpha) \circ i_\alpha]^{-1}$ provide a local trivialization

$\{\psi_{U_\alpha}\}_{\alpha \in \Lambda}$ of $P \xrightarrow{\pi} M$ which is compatible with the right action of G on P . Since ψ_{U_α} is a bijection for each α we can define a differentiable structure on $\pi^{-1}(U_\alpha)$ by forcing ψ_{U_α} to be a diffeomorphism.

Thus charts in this structure are composites of ψ_{U_α} with charts of $U_\alpha \times G$. Thus each subset $\pi^{-1}(U_\alpha)$ of P is a manifold. We claim that there exists

a unique structure on P such that $\pi^{-1}(U_\alpha)$ is an open submanifold of P for each α . To see this

first observe that since $M = \bigcup_\alpha U_\alpha$ we have

$P = \bigcup_\alpha \pi^{-1}(U_\alpha)$. If $\pi^{-1}(U_\alpha)$ is to be an open

submanifold of P for each α then the

structures on $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$ induced by

ψ_{U_α} and ψ_{U_β} must be compatible; to show

this we have only to show that there exists

a diffeomorphism $\varphi : (U_\alpha \cap U_\beta) \times G \rightarrow (U_\alpha \cap U_\beta) \times G$ such that

$$[\psi_{U_\alpha} | \pi^{-1}(U_\alpha \cap U_\beta)] = \varphi \circ [\psi_{U_\beta} | \pi^{-1}(U_\alpha \cap U_\beta)]$$

So let $u \in \pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta) = \pi^{-1}(U_\alpha \cap U_\beta)$

and observe that $u = [B, x, g]$ for some $x \in U_\alpha \cap U_\beta$ and $g \in G$. We have

$$\psi_{U_\beta}(u) = i_\beta^{-1}((\eta|S_\beta)^{-1}(B, x, g)) = i_\beta^{-1}(B, x, g)$$

$$= (x, g) = i_\alpha^{-1}(\alpha, x, g)$$

$$= i_\alpha^{-1}((\eta|S_\alpha)^{-1}(\alpha, x, g))$$

$$= \psi_{U_\alpha}([\alpha, x, g]) = \psi_{U_\alpha}([B, x, g_{\beta\alpha}(x)g])$$

$$= \psi_{U_\alpha}(u)(g_{\beta\alpha}^{-1}g(x)g)$$

where we used the fact that both $(\eta|S_\alpha)^{-1}$ and i_α^{-1} both commute with the group actions on their respective domains. If $\varphi : (U_\alpha \cap U_\beta) \times G \rightarrow (U_\alpha \cap U_\beta) \times G$ is defined by $\varphi(w) = w(g^{-1}g(x)g)$ we see that φ is a diffeomorphism and that

$$\psi_{U_\beta}(u) = \varphi(\psi_{U_\alpha}(u))$$

for $u \in \pi^{-1}(U_\alpha \cap U_\beta)$. So $\pi^{-1}(U_\alpha) \times P$ has a differentiable structure such that $\pi^{-1}(U_\alpha)$ is an open submanifold of P for each α and $\psi_{U_\alpha} : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$ is a diffeomorphism such that $\pi_\alpha \circ \psi_{U_\alpha} = \pi$.

Since $\psi_{U_\alpha}(u \cdot a) = \psi_{U_\alpha}(u)a$ for $u \in \pi^{-1}(U_\alpha)$, $a \in G$

we have that $\{\psi_{U_\alpha}\}$ is a local trivialization for a G -bundle structure on $\pi : P \rightarrow M$

Finally observe that the equation above shows that for $u \in \pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$

$$\psi_{U_\beta}(ug^{-1}) = \psi_{U_\alpha}(ug^{-1})g_{\beta\alpha}(x)$$

for all $g \in G$. So for $a = g$, $\psi_{U_\beta}(u) = \psi_{U_\alpha}(u)g_{\beta\alpha}(x)$

transition functions for the trivialization $\{\psi_{\alpha}\}$
The theorem follows.

Because of the last theorem we see that although the notion of a family of transition functions is derived from some local trivialization it is in fact characterized by the cocycle condition which makes no mention of a local trivialization. We formalize the idea by a definition.

Definition Assume $\pi: P \rightarrow M$ is a principal G -bundle and that $\{U_\alpha\}_{\alpha \in I}$ is an open cover of M . A family of mappings $\{g_{\beta\alpha}\}$, $g_{\beta\alpha}: U_\alpha \cap U_\beta \rightarrow G$, defined for α, β such that $U_\alpha \cap U_\beta \neq \emptyset$, is called a family of transition functions of the bundle P iff it satisfies the condition that if α, β, γ satisfy $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ then

$$g_{\gamma\alpha} = g_{\gamma\beta} g_{\beta\alpha}.$$

This latter condition will be referred to as the cocycle condition.

Definition Assume that $\pi: P \rightarrow M$ is a principal G -bundle and that H is a Lie subgroup of G . A principal H -bundle $\tau: Q \rightarrow M$ is called a reduction of $\pi: P \rightarrow M$ iff Q is a submanifold of P and $\tau = \pi|_Q$. In this case we say Q is the reduced bundle or that π reduces to τ . Also we say that π is reducible

H-bundle $\tau: Q \rightarrow M$, s.t. $N \subset M$ is a submanifold of M such that $Q = \pi^{-1}(N)$ is a submanifold of P then $\pi|_Q: Q \rightarrow N$ is a principal G -bundle which we refer to as the restriction of P to N . In this case Q is called the restricted bundle. Restricted bundles and bundle reductions are special cases of what are subbundles of $\pi: P \rightarrow M$ which will be defined more fully later in the exposition. In the meantime we may refer to either of these as subbundles of $\pi: P \rightarrow M$.

Theorem Given a principal G -bundle $\pi: P \rightarrow M$ and a Lie subgroup H of G , $\pi: P \rightarrow M$ is reducible to H iff there exists an open cover $\{U_\alpha\}$ of M and a family of transition functions $\{g_{\beta\alpha}\}$ of π which take their values in H .

Proof First assume that $\pi: P \rightarrow M$ is reducible and let $\tau: Q \rightarrow M$ denote the bundle to which it reduces. Let $\{\psi_\alpha\}_{\alpha \in U}$ be a local trivialization of τ as a principal H -bundle. Recall that if $U = \{U_\alpha\}_{\alpha \in A}$ and $\psi_\alpha = \psi_{\alpha\beta}$ for each $\alpha \in A$ then the transition functions $\{h_{\beta\alpha}\}$ of the local trivialization $\{\psi_\alpha\}$ are defined by

$$h_{\beta\alpha}(x) = \pi_H((\psi_\beta \circ \psi_\alpha^{-1})(x, e))$$

for $x \in U_\alpha \cap U_\beta$, where $\pi_H: (U_\alpha \cap U_\beta) \times H \rightarrow H$ is an ordinary projection map. Observe that if

$s_\alpha: U_\alpha \rightarrow \tilde{\pi}^{-1}(U_\alpha) \subset \tilde{\pi}(U_\alpha)$ is defined by $s_\alpha(x) = \psi_{\alpha, \pi(x)}$.
 for $x \in U_\alpha$, then s_α is a section of both

$$\pi^{-1}(U_\alpha) \xrightarrow{\pi} U_\alpha \text{ and } \tilde{\pi}^{-1}(U_\alpha) \xrightarrow{\tilde{\pi}} U_\alpha. \text{ Moreover}$$

$$\psi_\alpha(s_\alpha(x)h) = \psi_{\alpha, \pi(x)}(h) = (x, h) \text{ for all } x \in U_\alpha, h \in H.$$

Since $\pi^{-1}(U_\alpha) = s_\alpha(U_\alpha) \cdot G$ this suggests that

we may define a local trivializing mapping

$$\tilde{\Phi}_\alpha: \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G \text{ by } \tilde{\Phi}_\alpha(s_\alpha(x)g) = (x, g).$$

$$\text{It is easy to show that } \tilde{\Phi}_\alpha(u \cdot g) = \tilde{\Phi}_\alpha(u)g$$

for $u \in \pi^{-1}(U_\alpha)$, $g \in G$. The inverse of $\tilde{\Phi}_\alpha$ clearly exists and is defined by $(x, g) \mapsto s_\alpha(x)g$. It is not difficult to show that $\tilde{\Phi}_\alpha^{-1}$ is smooth and

by use of the inverse function theorem to show

$$\text{that } \tilde{\Phi}_\alpha \text{ itself is smooth. Since } \pi_{U_\alpha}(\tilde{\Phi}_\alpha^{-1}) = x = u$$

for $u = s_\alpha(x)g \in s_\alpha(U_\alpha) \cdot G = \pi^{-1}(U_\alpha)$ we see that $\tilde{\Phi}_\alpha$ satisfies all the properties required of a local

trivializing mapping of a principal bundle. We

now show that the transition functions of
 the local trivialization $\{\tilde{\Phi}_\alpha\}$ have their values
 in H . By the definition of $\{h_{\beta\alpha}\}$ given above we see

that

$$(\psi_\beta \circ \tilde{\Phi}_\alpha^{-1})(x, e) = (x, h_{\beta\alpha}(x));$$

consequently $\psi_\beta(s_\alpha(x)) = (x, h_{\beta\alpha}(x))$ for $x \in U_\alpha \cap U_\beta$.

Let $\{g_{\beta\alpha}\}$ denote the transition functions of $\{\tilde{\Phi}_\alpha\}$ and observe that the identity $\tilde{\Phi}_\alpha(s_\alpha(x)g) = (x, g)$ implies that $\tilde{\Phi}_\alpha|_{\tilde{\pi}^{-1}(U_\alpha)} = \Phi_\alpha$. Moreover it also tells us

that $\tilde{\Phi}_\alpha(s_\alpha(x)) = (x, e)$ and hence that $s_\alpha(x) = \tilde{\Phi}_\alpha^{-1}(x, e)$.

It follows that $g_{\beta\alpha}(x) = \pi_G((\psi_\beta \circ \tilde{\Phi}_\alpha^{-1})(x, e)) = \pi_G(\psi_\beta(s_\alpha(x)))$
 $= \pi_G(\psi_\beta(s_\alpha(x))) = \pi_G(x, e) = h_{\beta\alpha}(x)$, for $x \in U_\alpha \cap U_\beta$.

So $\{g_{\beta\alpha}\}$ are transition functions of $\pi: P \rightarrow M$ with values in H , as required.

Conversely, assume that there is a local trivialization $\{\tilde{\tau}_{\alpha}\}_{\alpha \in A}$ of the principal G -bundle $\pi: P \rightarrow M$ having transition functions $\{h_{\beta\alpha}\}$ with values in a Lie subgroup H of G . We show π is reducible to H . Let $\mathcal{U} = \{U_{\alpha}\}_{\alpha \in A}$ and $\tilde{P}_{\alpha} = \tilde{\tau}_{\alpha}^{-1}(U_{\alpha})$. Since $\{h_{\beta\alpha}\}$ are transition functions we know they satisfy the cocycle property and by the last theorem there is a principal H -bundle $\tau: Q \rightarrow M$ with transition functions $\{h_{\beta\alpha}\}$. Then there exists local trivializing functions

$\psi_{\alpha}: \tau^{-1}(U_{\alpha}) \rightarrow U_{\alpha} \times H$ of τ with transition functions $\{h_{\beta\alpha}\}$. For each $\alpha \in A$ let

$$f_{\alpha} = \tilde{P}_{\alpha}^{-1} \circ i_{\alpha} \circ \psi_{\alpha}$$

where $i_{\alpha}: U_{\alpha} \times H \rightarrow U_{\alpha} \times G$ is the inclusion mapping. Observe that f_{α} is a mapping from $\tau^{-1}(U_{\alpha})$ to $\pi^{-1}(U_{\alpha})$ which is clearly smooth and injective. Moreover, $f_{\alpha}(u)h = f_{\alpha}(u)h$ for $u \in \tau^{-1}(U_{\alpha})$, $h \in H$ and it follows from the commutative diagram:

$$\begin{array}{ccccccc} \tau^{-1}(U_{\alpha}) & \xrightarrow{\psi_{\alpha}} & U_{\alpha} \times H & \xrightarrow{i_{\alpha}} & U_{\alpha} \times G & \xrightarrow{\tilde{P}_{\alpha}^{-1}} & \pi^{-1}(U_{\alpha}) \\ & \searrow \tau & \downarrow & \downarrow & \swarrow \pi & & \\ & & U_{\alpha} & & & & \end{array}$$

that $\pi \circ f_{\alpha} = \tau$. Since ψ_{α} and \tilde{P}_{α} are

diffeomorphisms it is obvious that $f_\alpha(\tilde{\tau}_\alpha^{-1}(U_\alpha))$ is a submanifold of $\pi^{-1}(U_\alpha)$ and that

$f_\alpha : \tilde{\tau}^{-1}(U_\alpha) \rightarrow f_\alpha(\tilde{\tau}_\alpha^{-1}(U_\alpha))$ is a diffeomorphism.

Thus one has an embedding of the principal

H -bundle $\tilde{\tau}^{-1}(U_\alpha) \rightarrow U_\alpha$ into $\pi^{-1}(U_\alpha) \rightarrow U_\alpha$

and $\pi^{-1}(U_\alpha) \rightarrow U_\alpha$ reduces to this H -subbundle.

If we show that for α, β such that $U_\alpha \cap U_\beta \neq \emptyset$

$$f_\alpha|_{\tilde{\tau}^{-1}(U_\alpha \cap U_\beta)} = f_\beta|_{\tilde{\tau}^{-1}(U_\alpha \cap U_\beta)}, \text{ then}$$

we may define $f : Q \rightarrow P$ by

$$f|_{\tilde{\tau}^{-1}(U_\alpha)} = f_\alpha \text{ for each } \alpha \in \Lambda \text{ and we}$$

will have an embedding of Q into P

with $f(Q) \subseteq P$ being an H -subbundle

of P to which P reduces. Thus we have only
to show that f_α and f_β agree on $\tilde{\tau}^{-1}(U_\alpha \cap U_\beta)$.

To prove this we first show that $(\psi_\beta \circ \tilde{\psi}_\alpha^{-1})(x, e) = (\tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1})(x, e)$
for all $x \in U_\alpha \cap U_\beta$. Since $h_{\beta\alpha}(x) = \pi_H((\psi_\beta \circ \tilde{\psi}_\alpha^{-1})(x, e))$
and $h_{\beta\alpha}(x) = \pi_G((\tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1})(x, e))$ we have

$$(\psi_\beta \circ \tilde{\psi}_\alpha^{-1})(x, e) = (x, h_{\beta\alpha}(x)) = (\tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1})(x, e) \text{ as}$$

required. Remember that both $\psi_\alpha, \psi_\beta, \tilde{\psi}_\alpha, \tilde{\psi}_\beta$
commute with the action of H on Q and P
respectively so that for $x \in U_\alpha \cap U_\beta$, $h \in H$

$$\begin{aligned} (\psi_\beta \circ \tilde{\psi}_\alpha^{-1})(x, h) &= (\tilde{\psi}_\beta^{-1} \circ i_\beta \circ \psi_\beta \circ \tilde{\psi}_\alpha^{-1})(x, h) \\ &= (\tilde{\psi}_\beta^{-1} \circ i_\beta \circ \psi_\beta \circ \tilde{\psi}_\alpha^{-1})(x, e) \cdot h \\ &= (\tilde{\psi}_\beta^{-1} \circ i_\beta \circ \tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1})(x, e) \cdot h \\ &= \tilde{\psi}_\alpha^{-1}(x, e) \cdot h \\ &= \tilde{\psi}_\alpha^{-1}(x, h). \end{aligned}$$

Thus $f_\beta \circ \psi_\alpha^{-1} = \psi_\alpha^{-1} \circ i_\alpha$ on $(U_\alpha \cap U_\beta) \times H$ and $f_\beta|_{\tilde{\iota}^{-1}(U_\alpha \cap U_\beta)} = f_\alpha|_{\tilde{\iota}^{-1}(U_\alpha \cap U_\beta)}$. This completes the proof of the theorem.

In many of the theories devised by physicists one has physical states which seemingly can not be directly formulated in terms of space-time concepts. The tangent bundle and various tensor bundles of space-time apparently are not adequate to describe all the physical states of the theory. In many such theories the space of states may be described using the language of vector bundles over space-time. At each point x of space-time M one has a vector space E_x which "varies smoothly" as a function of x and E_x represents the space of all states of the theory localized at x . The vector space is referred to as the space of internal states of the theory. For example the simplest theory of isospin utilizes a vector bundle E over Minkowski space M in which E_x is a complex 2-dimensional vector space. Usually these bundles are trivial so that $E = M \times \mathbb{C}^2$ and for each x , $E_x = \{x\} \times \mathbb{C}^2$ is identified with \mathbb{C}^2 . Moreover there is usually a preferred reference frame of E_x at each $x \in M$. In the isospin case the

frame of E_x is $(x, e_1), (x, e_2)$ where $e_1 = (1, 0), e_2 = (0, 1)$. This frame is utilized without explicit mention of the fact. Moreover this particular theory, as well as others devised by physicists, employ what one calls a "fiber metric". In the isospin case the metric is complex, $g_x : E_x \times E_x \rightarrow \mathbb{C}$ is defined by

$$g_x(z, w) = z_1 \bar{w}_1 + z_2 \bar{w}_2$$

where $x \in M$ and $z, w \in E_x \cong \mathbb{C}^2$. Occasionally the preferred reference frame is changed to some other g_x -orthonormal frame at each $x \in M$. In mathematical language we would describe these choices and changes of choices as follows.

Let $O_x E$ denote the set of all pairs $(x, \{f_i\})$ such that $x \in M$ and $\{f_i\}$ is a g_x -orthonormal basis of E_x . Let $\pi : O_x E \rightarrow M$ denote

the mapping defined by $\pi(x, \{f_i\}) = x$. Then $O_x E \xrightarrow{\pi} M$ is a principal fiber bundle with structure group the set of all g_x -isometries.

In the isospin case the group is $SO(2)$. The preferred reference frame is a section of this bundle, in the Isospin case

$$s(x) = (x, \{e_i\})$$

where $e_1 = (1, 0), e_2 = (0, 1)$. One could call such a section a gauge of the theory. If one changes reference frames at each point one has defined another sect. $\tau : M \rightarrow O_x E$ of

the principal bundle. Notice that at each $x \in M$ there is a unique $A(x) \in SO(2)$ which "rotates" the frame at x defined by $s(x)$ to the frame at x defined by $\bar{s}(x)$. Thus

$$\bar{s}(x) = s(x) \cdot A(x)$$

for each $x \in M$. The mapping $x \mapsto A(x)$ clearly transforms the gauge s to the gauge \bar{s} and so is called a gauge transformation. Since these ideas are so useful in describing physical theories we now show how these ideas relate to transition functions.

The remarks in the last paragraph were formulated over Minkowski space-time rather than a more general space-time for simplicity. In case M is Minkowski space-time all fiber bundles over M are trivial and the sections utilized may be defined on all of M . For more general space-times only local sections exist and so one may only choose reference frames smoothly throughout an open subset of M in the case of a bundle of frames over M .

Lemma If $\pi: P \rightarrow M$ is a principal G -bundle and $U \subseteq M$ is open, then there exists a section $s: U \rightarrow \pi^{-1}(U)$ of the principal bundle $\pi|_{\pi^{-1}(U)}: \pi^{-1}(U) \rightarrow U$ iff there is a trivializing mapping $\psi: \pi^{-1}(U) \rightarrow U \times G$ of $\pi|_{\pi^{-1}(U)}$.

Proof. First assume that $\psi: \pi^{-1}(U) \rightarrow U \times G$ is a trivializing mapping. Define $s: U \rightarrow \pi^{-1}(U)$ by $s(x) = \psi^{-1}(x, e)$. Clearly s is smooth and since $\psi(s(x)) = (x, e)$ we have $\pi_U(\psi(s(x))) = x$ and $\pi(\psi(s(x))) = x$. Thus $s: U \rightarrow \pi^{-1}(U)$ is a section of $\pi|_{\pi^{-1}(U)}$.

Conversely, if $U \subseteq M$ is open and $s: U \rightarrow \pi^{-1}(U)$ is a section of $\pi|_{\pi^{-1}(U)}$ then we may define a mapping $\Phi: U \times G \rightarrow \pi^{-1}(U)$ by $\Phi(x, g) = s(x) \cdot g = s(xg)$ where \cdot is the action of G on the principal bundle P . Thus Φ is smooth. It is easy to check that Φ is bijective using the fact that \cdot is free. The inverse function theorem guarantees that Φ^{-1} is smooth. Moreover $(\pi \circ \Phi)(x, g) = \pi(s(x) \cdot g) = \pi(s(x)) = x$. It is now easy to show that $\psi = \Phi^{-1}$ is a local trivializing mapping from $\pi^{-1}(U)$ onto $U \times G$. Observe that the trivializing mapping ψ respects the group action as it should since $\Phi(x, g_1)g_2 = \Phi(x, gg_2) = s(x)g_1g_2 = \Phi(x, g_1)g_2$ for $x \in U$, $g_1, g_2 \in G$.

~~Corollary~~

Theorem. Let $\pi: P \rightarrow M$ be a principal G -bundle and let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of M . Then there exists local trivializing mappings $\psi_\alpha: \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$ defined over the

cover $\{U_\alpha\}$ iff there exist local sections

$s_\alpha : U_\alpha \rightarrow \pi^{-1}(U_\alpha)$ defined over the same open cover. Moreover, if one has both $\{\psi_\alpha\}$ and $\{s_\alpha\}$ for a given cover $\{U_\alpha\}$ and if $\{g_{\beta\alpha}\}$ are transition mappings of $\{\psi_\alpha\}$ then $s_\alpha(x) = s_\beta(x) g_\beta(x)^{-1} g_\alpha(x) g_\beta(x)$ where $g_\gamma(y) = \pi_G(\psi_\gamma(s_\gamma(y)))$ for $y \in U_\gamma$ and $x \in U_\alpha \cap U_\beta$.

Proof The first assertion of the Theorem is an immediate consequence of the lemma. We prove the assertion regarding the transition functions $\{g_{\beta\alpha}\}$. Recall that the transition functions are defined by

$$g_{\beta\alpha}(x) = \pi_G(\psi_\beta(s_\alpha(x)))$$

for all $x \in U_\alpha \cap U_\beta$ and $g_\gamma(x) = \pi_G(\psi_\gamma(s_\gamma(x)))$ for all $x \in U_\gamma$. Thus $\psi_\gamma(s_\gamma(x)) = (x, g_\gamma(x))$ for $x \in U_\gamma$ and $s_\gamma(x) = \psi_\gamma^{-1}(x, g_\gamma(x))$. It follows that

$$\begin{aligned} \psi_\beta(s_\alpha(x)) &= \psi_\beta(\psi_\alpha^{-1}(x, g_\alpha(x))) \\ &= \psi_\beta(\psi_\alpha^{-1}(x, e)) g_\alpha(x) \\ &= (x, g_{\beta\alpha}(x)) g_\alpha(x) \end{aligned}$$

and $s_\alpha(x) = \psi_\beta^{-1}(x, g_{\beta\alpha}(x)) g_\alpha(x)$

$$\begin{aligned} &= \psi_\beta^{-1}(x, e) g_{\beta\alpha}(x) g_\alpha(x) \\ &= \psi_\beta^{-1}(x, g_\beta(x)) g_\beta(x)^{-1} g_{\beta\alpha}(x) g_\alpha(x) \\ &= s_\beta(x) g_\beta(x)^{-1} g_{\beta\alpha}(x) g_\alpha(x) \end{aligned}$$

as asserted.