

In Physics bundle reduction is ^{often} attained via the "level surfaces" of special kinds of functions called "equivariant mappings". After defining the relevant concept and clarifying it with examples we prove a theorem which gives conditions under which bundle reduction is implemented via such functions.

Definition Let $\pi: P \rightarrow M$ denote a principal G -bundle and assume that G acts on the left of some manifold F . A smooth mapping $\varphi: P \rightarrow F$ is equivariant iff $\varphi(ug) = g^{-1} \cdot \varphi(u)$ for all $u \in P, g \in G$. In the case the action of G on F is a right action the equivariance condition is that $\varphi(ug) = \varphi(u) \cdot g$ for $u \in P, g \in G$.

Remark. Recall that every finite-dimensional vector space V has a natural manifold structure. In many applications the manifold F of the last definition is just a vector space V . In such cases the group often acts via linear mappings of V , i.e., for a right action $\tau_g: V \rightarrow V$ is linear for each $g \in G$. Consequently, the action is given by a linear representation $g \mapsto \tau_g$ of the group G on some vector space V .

In other applications F is realized as a manifold G/H i.e., H is a normal subgroup

in such case

subgroup of G . In fact if the action of G on F is transitive and $x \in F$ then F can be identified with G/G_x where G_x is the isotropy subgroup of x and the action of G on F may be identified with the action of G on G/G_x defined by

$$(g_1 g_2 G_x) \mapsto (g_1 g_2) G_x, \quad g_1, g_2 \in G.$$

Example 1 Let $\pi: E \rightarrow M$ denote any vector bundle with fiber V and let $\tau: \mathcal{F}E \rightarrow M$ denote the bundle of frames of E . Let $\{\hat{r}_i \mid 1 \leq i \leq n\}$ denote a basis of V and assume that G acts linearly on V . For each $g \in G$ define a matrix (g_{ij}^g) via the equation $g \cdot \hat{r}_j = g_{ij}^g \hat{r}_i$. We identify G with $GL(n)$ by identifying $g \in G$ with its matrix (g_{ij}^g) . We show that each section $\varphi: M \rightarrow E$ of π defines an equivariant mapping $\hat{\varphi}: \mathcal{F}E \rightarrow V$. Actually the converse is also true but this fact and its implications will be explored in more detail later. Given a section $\varphi: M \rightarrow E$ we have that $\varphi(x) \in E_x$ and consequently if $\{e_i\}$ is a basis of E_x and $\{e^i\}$ is the basis of E_x^* dual to $\{e_i\}$ then $e^i(\varphi(x)) \hat{r}_i \in V$. Define $\hat{\varphi}: \mathcal{F}E \rightarrow V$ by

$$\hat{\varphi}(x, \{e_i\}) = e^i(\varphi(x)) \hat{r}_i.$$

We show $\hat{\varphi}$ is equivariant. Recall that $G = GL(n)$

and if $f_i = e_j g_j^i$, then the basis of E_x^* dual to $\{f_i\}$ is $f^i = (g^{-1})_i^j e^j$ and

$$\begin{aligned} \hat{\varphi}(x, \{e_i\} \circ g) &= \hat{\varphi}(x, \{f_i\}) = f^i(\varphi(x)) \hat{h}_i \\ &= (g^{-1})_j^i e^j(\varphi(x)) \hat{h}_i \\ &= e^j(\varphi(x)) (g^{-1})_j^i \hat{h}_i \\ &= g^{-1} \cdot [e^j(\varphi(x)) \hat{h}_j] \\ &= g^{-1} \circ \hat{\varphi}(x, \{e_i\}). \end{aligned}$$

Thus $\hat{\varphi}$ is equivariant.

This example is related to scalar fields in Physics as follows. Let $\tilde{\varphi}: M \rightarrow \mathbb{C}^N$ be a scalar field on Minkowski space M . Such fields are in one-to-one correspondence with mappings $\varphi: M \rightarrow M \times \mathbb{C}^N$ defined by $\varphi(x) = (x, \tilde{\varphi}(x))$, $x \in M$. Now $E = M \times \mathbb{C}^N$ is the bundle space of the trivial vector bundle $\pi: E \rightarrow M$ defined by $\pi(x, z) = x$ for $(x, z) \in M \times \mathbb{C}^N$. Thus fields $\tilde{\varphi}: M \rightarrow \mathbb{C}^N$ define sections of $\pi: E \rightarrow M$ and conversely. On the other hand each section $\varphi: M \rightarrow E$ defines an equivariant mapping $\hat{\varphi}: \mathcal{F}E \rightarrow \mathbb{C}^N$ as shown in the example above. What does $\hat{\varphi}$ encode which is not already obviously encoded into $\tilde{\varphi}$? The answer is that $\tilde{\varphi}$ defines the field relative to a single reference frame in \mathbb{C}^N .

which is not even explicitly defined whereas $\hat{\varphi}$ defines $\tilde{\varphi}$ in all possible reference frames. To see this recall that a section $s: M \rightarrow FE$ of $\pi: FE \rightarrow M$ defines a reference frame of E_x at each point x of M . Indeed if $s(x) = (x, \{e_i(x)\})$, for $x \in M$, then $\{e_i(x)\}$ is a basis of E_x . Moreover $s^* \hat{\varphi}$ is then a mapping from M to \mathbb{C}^N which can be compared to the original mapping $\tilde{\varphi}$. If we identify E_x with $\{x\} \times \mathbb{C}^N$ and $\hat{\varphi}_i$ with $(x, \hat{\varphi}_i)$ where $\{\hat{\varphi}_i\}$ is the standard basis of \mathbb{C}^N , then the pullback of $\hat{\varphi}$ under the special section $s(x) = (x, \{\hat{\varphi}_i\})$ is simply $\tilde{\varphi}$. Indeed

$$\begin{aligned} (s^* \hat{\varphi})(x) &= \hat{\varphi}(s(x)) = \hat{\varphi}(x, \{\hat{\varphi}_i\}) = \hat{\varphi}^i(x) \hat{\varphi}_i \\ &= \hat{\varphi}^i(x, \tilde{\varphi}^k(x) \hat{\varphi}_k) = \tilde{\varphi}^i(x) \hat{\varphi}_i = \tilde{\varphi}(x) \end{aligned}$$

for $x \in M$. Thus $\tilde{\varphi}$ is simply the pullback of $\hat{\varphi}$ under a section which chooses the standard basis of $E_x \approx \mathbb{C}^N$ at each space-time point $x \in M$. The pullback of $\hat{\varphi}$ relative to other sections gives the components of $\tilde{\varphi}$ relative to other choices of reference frames at points of M .

Example Let (M, g) be a Riemannian manifold of dimension m . Let $\hat{g}: FM \rightarrow T_2^0 \mathbb{R}^m$ be defined by $\hat{g}(x, \{e_i\}) = g_x(e_i, e_j) (\nu^i \otimes \nu^j)$, where $\{\nu_i\}$ is the standard basis of \mathbb{R}^m and $\{\nu^i\}$ is the basis of $(\mathbb{R}^m)^*$ dual to $\{\nu_i\}$. We show

that \hat{g} is equivariant relative to the left action of $GL(n)$ on $T_2^0 \mathbb{R}^m$ defined by

$$g \cdot (r^i \otimes r^k) = (g \cdot r^i) \otimes (g \cdot r^k) = (g^{-1})^i{}_k (g^{-1})^k{}_l (r^l \otimes r^l)$$

for $g = (g^i{}_j) \in GL(n)$. For $(x, \{e_i\}) \in FM$, $a \in GL(n)$,

$$\begin{aligned} \hat{g}((x, \{e_i\}) \cdot a) &= \hat{g}(x, \{e_j a^j{}_i\}) \\ &= g_x(e_i a^i{}_k, e_j a^j{}_l) (r^k \otimes r^l) \\ &= a^i{}_k a^j{}_l g_x(e_i, e_j) (r^k \otimes r^l) \\ &= g_x(e_i, e_j) [(a^{-1})^i{}_k (a^{-1})^j{}_l] \\ &= a^{-1} \cdot [g_x(e_i, e_j) (r^i \otimes r^j)] \\ &= a^{-1} \cdot \hat{g}(x, \{e_i\}). \end{aligned}$$

So \hat{g} is equivariant.

This example shows how metrics on a space-time manifold may be reformulated as equivariant maps on the principal fiber bundle $FM \rightarrow M$. The most basic fields of gravitational theories are those defined by metrics. The latter example tells us that these fields become equivariant maps on FM . We will eventually see that each of the basic fields which occur in Lagrange field theory may be formulated on some principal bundle. In many instances, as in the Examples above, the principal bundle encodes information relating to choices of reference frames. In other

instances, such as gauge theory, symmetries of the theory are implicitly encoded via the structure group of the principal bundle. In general, principal bundles provide a unifying framework by which one may understand the geometric structure underlying most field theories of physics.

We now prove a sequence of Lemmas whose goal is to show how equivariant mappings may be used to obtain reductions of bundles.

Lemma. Let M and F denote manifolds and G a Lie group which acts transitively on the left of F . Let $\pi: M \times G \rightarrow M$ denote the trivial principal G -bundle and assume that $\varphi: M \times G \rightarrow F$ is an equivariant mapping. If $y_0 \in F$, then for each $x_0 \in M$ there exists an open subset U_0 of M containing x_0 and a section $s: U_0 \rightarrow U_0 \times G$ of $\pi|_{(U_0 \times G)}$ such that $\varphi(s(x)) = y_0$ for all $x \in U_0$.

Proof. Since the action of G on F is transitive we see that $F = G \cdot y_0 = G/G_{y_0}$ where G_{y_0} is the isotropy subgroup of $y_0 \in F$. Moreover the mapping $\tau: G \rightarrow G/G_{y_0} = F$ defined by $\tau(g) = g \cdot y_0$ is a principal G_{y_0} -bundle. If $e^2 = e$ is the identity of G , then $\varphi(x_0, e) \in F$ and there is a local section $\sigma: W_0 \rightarrow G$ of τ such that $\tau(\sigma(x)) = e$ for all $x \in W_0$. Let $U_0 = \pi^{-1}(W_0) \cap (U_0 \times G)$ and define $s: U_0 \rightarrow U_0 \times G$ by $s(x) = (x, \sigma(x))$. Then $\varphi(s(x)) = \varphi(x, \sigma(x)) = \sigma(x) \cdot y_0 = e \cdot y_0 = y_0$ for all $x \in U_0$.

defined by $\varphi_e(x) = \varphi(x, e)$. Then $U_0 = \varphi_e^{-1}(W_0)$ is open in M and contains x_0 . Let $g: U_0 \rightarrow G$ be defined by $g(x) = \sigma(\varphi(x, e))$ for $x \in U_0$. Since $\tau \circ \sigma = \text{id}_{W_0}$, $\tau(g(x)) = \varphi(x, e)$ for all $x \in U_0$. Thus $g(x) \cdot y_0 = \varphi(x, e)$ and $y_0 = g(x)^{-1} \cdot \varphi(x, e) = \varphi(x, g(x))$. If we let $\Delta(x) = (x, g(x))$ we see that Δ is a section of $U_0 \times G \rightarrow U_0$ and $\varphi(\Delta(x)) = y_0$ for each $x \in U_0$.

Lemma Assume that $\pi: P \rightarrow M$ is a principal G -bundle and that G acts transitively on the left of some manifold F . If $\varphi: P \rightarrow F$ is equivariant, $y_0 \in F$, and $x_0 \in M$, then there exists an open subset U_0 of M containing x_0 and a section $\Delta: U_0 \rightarrow \pi^{-1}(U_0)$ such that $\varphi(\Delta(x)) = y_0$ for all $x \in U_0$.

Proof Choose U open about x_0 in M such that there exists a local trivializing mapping ψ from $\pi^{-1}(U)$ onto $U \times G$. Define $\varphi_\psi: U \times G \rightarrow F$ by $\varphi_\psi = \varphi \circ \psi^{-1}$. Now

$$\varphi_\psi(x, g \cdot a) = \varphi_\psi(x, ga) = \varphi(\psi^{-1}(x, g \cdot a)) = a^{-1} \cdot \varphi_\psi(a, g)$$

for $x \in U$, $a, g \in G$. Thus φ_ψ is equivariant. Now $x_0 \in U$ so by the previous lemma there exists U_0 open about x_0 in U and a section $\Delta_0: U_0 \rightarrow U_0 \times G$ of the trivial bundle $U_0 \times G \rightarrow U_0$ such that

Let $s: U_0 \rightarrow \pi^{-1}(U_0)$ is defined by $s(x) = \psi^{-1}(s_0(x))$
 we see from the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & U \times G \\ & \searrow \pi & \swarrow \pi_U \\ & U & \end{array}$$

that $\pi(s(x)) = \pi(\psi^{-1}(s_0(x))) = \pi_U(s_0(x)) = x$ for $x \in U_0$
 and thus s is a section of $\pi^{-1}(U_0) \rightarrow U_0$ such
 that $\varphi(s(x)) = \varphi(\psi^{-1}(s_0(x))) = \varphi_\psi(s_0(x)) = y_0$ for all $x \in U_0$.
 The lemma follows.

Theorem Assume $\pi: P \rightarrow M$ is a principal G -bundle
 and that G acts transitively on the left of a
 manifold F . Let $\varphi: P \rightarrow F$ is equivariant
 and $y_0 \in F$ then there exists an open cover
 $\{U_\alpha\}_{\alpha \in \Lambda}$ of M and a family $\{s_\alpha\}_{\alpha \in \Lambda}$ of local
 sections $s_\alpha: U_\alpha \rightarrow \pi^{-1}(U_\alpha)$ of π such that
 $\varphi(s_\alpha(x)) = y_0$ for all $x \in U_\alpha$.

Proof This theorem is an immediate consequence
 of the last lemma as it asserts that each
 point of M is contained in an open
 subset $U_0 \subseteq M$ over which there is a
 section $s_0: U_0 \rightarrow \pi^{-1}(U_0)$ such that
 $\varphi(s_0(x)) = y_0$ for all $x \in U_0$.

Corollary Assume $\pi: P \rightarrow M$ is a principal G -bundle and that G acts transitively on the left of some manifold F . If $\varphi: P \rightarrow F$ is an equivariant mapping, then $\pi: P \rightarrow M$ reduces to the G_y -subbundle $\varphi^{-1}(y) \subseteq P$ for each $y \in F$.

Proof. By the Theorem there exists an open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of M and local sections $s_\alpha: U_\alpha \rightarrow \pi^{-1}(U_\alpha)$ of π such that $\varphi(s_\alpha(x)) = c$ for all $x \in U_\alpha$. For each local section s_α define a mapping from $U_\alpha \times G$ to $\pi^{-1}(U_\alpha)$ by $(x, g) \mapsto s_\alpha(x)g$. This mapping is smooth, bijective, and has a smooth inverse (by the inverse function theorem). Its inverse, ψ_α , is a local trivializing mapping for the principal G -bundle π , in particular $\psi_\alpha(u \cdot a) = \psi_\alpha(u)a$ for $u \in \pi^{-1}(U_\alpha)$, $a \in G$. The proofs of these assertions are similar to the proof of Theorem — and are left to the reader. We claim that $\psi_\alpha|_{\varphi^{-1}(y)}$ is a local trivializing mapping from $\pi^{-1}(U_\alpha) \cap \varphi^{-1}(y)$ onto $U_\alpha \times G_y$ for each $y \in F$. To see this we first show that $\pi^{-1}(U_\alpha) \cap \varphi^{-1}(y) = \psi_\alpha^{-1}(U_\alpha \times G_y)$. Let $u \in \pi^{-1}(U_\alpha) \cap \varphi^{-1}(y)$, then u and $s(\pi(u))$ are both in the fiber $\pi^{-1}(\pi(u)) = u \cdot G$ thus $u = s(\pi(u))g$ for some $g \in G$. But $u \in \varphi^{-1}(y)$ implies $y = \varphi(u) = \varphi(s(\pi(u))g) = \varphi(s(\pi(u))) = \varphi(s(\pi(u)))$ and

consequently $a \in G_y$. Thus

$$u = s_x(\pi(u)) \cdot g = \psi_\alpha^{-1}(\pi(u), g) \in \psi_\alpha^{-1}(U_\alpha \times G_y)$$

and

$$\pi^{-1}(U_\alpha) \cap \bar{\varphi}^{-1}(y) \subseteq \psi_\alpha^{-1}(U_\alpha \times G_y).$$

Conversely $u \in \psi_\alpha^{-1}(U_\alpha \times G_y)$ implies that

$$u = \psi_\alpha^{-1}(x, g) \text{ for some } (x, g) \in U_\alpha \times G_y \text{ and}$$

$$u = s_x(x) \cdot g. \text{ Now } \pi(u) = \pi(s_x(x) \cdot g) = \pi(s_x(x)) = x$$

$$\text{and } \varphi(u) = \varphi(s_x(x) \cdot g) = g^{-1} \cdot \varphi(s_x(x)) = g^{-1} \cdot y = y.$$

Consequently $u \in \pi^{-1}(U_\alpha) \cap \bar{\varphi}^{-1}(y)$ and

$$\psi_\alpha^{-1}(U_\alpha \times G_y) \subseteq \pi^{-1}(U_\alpha) \cap \bar{\varphi}^{-1}(y).$$

Let $\tau = \pi|_{\bar{\varphi}^{-1}(y)}$. We see that

$$\tau^{-1}(U_\alpha) = \pi^{-1}(U_\alpha) \cap \bar{\varphi}^{-1}(y) = \psi_\alpha^{-1}(U_\alpha \times G_y).$$

So if $\psi_\alpha^\tau \equiv \psi_\alpha|_{\tau^{-1}(U_\alpha)}: \tau^{-1}(U_\alpha) \rightarrow U_\alpha \times G_y$

then ψ_α^τ is a diffeomorphism (since ψ_α is a diffeomorphism) and is clearly a trivializing mapping. It follows that $\{\psi_\alpha^\tau\}$ is a family of local trivializing mappings for the G_y -bundle $\tau: \bar{\varphi}^{-1}(y) \rightarrow M$. Clearly $\pi: P \rightarrow M$ reduces to the subbundle $\tau: \bar{\varphi}^{-1}(y) \rightarrow M$.

In Weinberg-Salam theory and in t' Hooft's version of a monopole there is an auxiliary field called a Higgs field. In Weinberg-Salam theory one begins with a field theory in which one has a family of fields whose quanta are massless bosons called Goldstone bosons. The Higgs field is introduced to provide a mechanism for "breaking the symmetry" of the theory.

This means that one introduces a field which interacts with the boson fields in such a manner that some of the bosons are eliminated but the remaining fields acquire mass.

In the context of bundle theory the Higgs field is an equivariant mapping from a principal $SU(2) \times U(1)$ bundle to \mathbb{C}^N . The field reduces the bundle via our last theorem to a $U(1)$ -subbundle where $U(1)$ is a nontrivial copy of $U(1)$ in $SU(2) \times U(1)$. Thus one trades the larger symmetry of the original fields represented by $SU(2) \times U(1)$ for a smaller symmetry $U(1)$ of the reduced bundle. The restriction of the bosonic fields to the smaller bundle results in some fields vanishing and others acquiring mass. For this procedure to work the Higgs field will be an equivariant mapping whose range lies in a single orbit of the action of $SU(2) \times U(1)$ on \mathbb{C}^N .

In a full theory the bosons which are eliminated become absorbed into the vacuum.

Let $\pi: B \rightarrow M$ denote a fiber bundle with fiber some manifold F . Many geometric structures on B are defined in terms of tangent vectors to B at various points of B . It is useful to be able to write such vectors $w \in T_u B$, $u \in B$, as the sum of two vectors $w_h + w_v$ where w_h is derived from a vector tangent to M at $\pi(u)$ and where w_v is, in some sense, tangent to the fiber F of the bundle. In case π is trivial one has that $B = M \times F$ and for $u = (x, f) \in B$, $T_u B = T_x M \oplus T_f F$ and this gives a decomposition of $T_u B$ such that each vector $w = w_h + w_v$ with $w_h \in T_x M$, $w_v \in T_f F$. In general B is locally trivial so that each $u \in B$ is in $\pi^{-1}(U)$ for some open set $U \subseteq M$ such that $\pi^{-1}(U)$ is bundle isomorphic to $U \times F$. If $\psi: \pi^{-1}(U) \rightarrow U \times F$ is a bundle isomorphism and $\pi_F: U \times F \rightarrow F$ is the usual projection, then $\text{diff}_u: T_u B \rightarrow T_{\pi(u)} U \oplus T_{\pi(u)} F$ is a vector space isomorphism. Since $T_p U = T_p M$ for each $p \in U$ we have a decomposition of $T_u B$ for each $u \in B$ but unfortunately if $u \in \pi^{-1}(U_1) \cap \pi^{-1}(U_2)$ for two distinct subbundles on which there are trivializing mappings $\psi_i: \pi^{-1}(U_i) \rightarrow U_i \times F$, $i = 1, 2$, then one does not have a unique decomposition of T_u . Notice that at each $u \in B$ there is a natural candidate for vectors in $T_u B$

which are in a sense tangent to the fiber F . The set of all vectors $V \in T_u B$ such that $d_u \pi(V) = 0$ are necessarily tangent to the submanifold $\pi^{-1}(\pi(u)) \subseteq B$. Since $\pi^{-1}(\pi(u))$ is diffeomorphic to F we see that each point $u \in B$ is in a submanifold $\pi^{-1}(\pi(u))$ which is a copy of F in B . The vectors tangent to $\pi^{-1}(\pi(u))$ at u are consequently tangent to a copy of F in B and are defined without recourse to a specific local trivialization. In a previous chapter we referred to the vectors of $T_u B$ which are tangent to $\pi^{-1}(\pi(u))$ as vertical vectors and we continue to use the notation developed there, namely

$$(VB)_u = V(T_u B) = \{v \in T_u B \mid d_u \pi(v) = 0\}.$$

Generally there is no "natural" choice of a complement of $V(T_u B)$ in $T_u B$ which depends only on the bundle structure. It can be proven that there are consistent ways to choose complements of $V(T_u B)$ in $T_u B$ which vary smoothly as functions of u but such a smoothly varying choice need not be unique. A single choice of this type is called a "connection". The precise definition follows.

Definition If $\pi: B \rightarrow M$ is a fiber bundle with fiber F then we say that H is an Ehresmann connection on B iff H is a mapping from B into TB such that

(1) for each $u \in B$, $H(u) = H(T_u B)$ is a subspace of $T_u B$ such that

$$T_u B = H(u) \oplus V(T_u B),$$

(2) the mapping H is smooth in the sense that for each vector field X on B the vector fields hX and vX , defined by

$$X(u) = (hX)(u) + (vX)(u), \quad (hX)(u) \in H(u), (vX)(u) \in V(T_u B)$$

for each $u \in B$, are smooth vector fields.

If $\pi: B \rightarrow M$ is a principal G -bundle then $F = G$ is a Lie group and H is called a principal connection or simply a connection iff it is an Ehresmann connection which satisfies the additional property that

(3) $d\pi_g(H(u)) = H(ug)$ for each $u \in B, g \in G$.

Observe that if $\pi: B \rightarrow M$ is any fiber bundle then the selection of a subspace $H(u) \subseteq T_u B$ complementary to $(VB)_u$ is completely equivalent to defining a projection $\gamma_u: T_u B \rightarrow T_u B$ such that

(i) the image of γ_u is $(VB)_u = V(T_u B)$, and

(ii) the kernel of γ_u is $H(u)$.

Moreover the smoothness of H is equivalent to saying that for each vector field X on B $\gamma(X)$ is smooth where $\gamma(X)$ is defined by

$$\gamma(X)(u) = \gamma_u(X(u))$$

for each $u \in B$. Indeed if γ is smooth in this sense then νX is smooth

since $(\nu X)(u) = \gamma_u(X(u))$ and moreover hX

is smooth since $(hX)(u) = X(u) - (\nu X)(u) =$

$X(u) - \gamma_u(X(u))$. Consequently if γ is smooth

then H is smooth. Conversely smoothness

of H trivially implies that for each vector field X , $\gamma(X)(u) = (\nu X)(u)$ is smooth.

In the case $\pi: B \rightarrow M$ is a principal G -bundle one may represent principal connections as Lie-algebra valued 1-forms on B . To see that such a representation should be possible note first that an Ehresmann connection γ on B may be regarded as a bundle-valued one-form on B . Indeed $\gamma(u)$ is a linear mapping from $T_u B$ into $V(T_u B)$ and so is a one-form on B with values in $V(TB)$, the bundle space of the vector bundle $V(TB) \rightarrow B$. It seems clear that it should be possible to identify the vertical vectors $V(T_u B)$ with tangents to the fiber G of π . But tangents to G may be identified with T_G which is the Lie-algebra of G .

The details are provided in the proof of the next theorem.

Recall that the Lie group G acts on the right of a principal G -bundle and for $A \in \mathfrak{g}$, the Lie algebra of G , δ_A is a vertical vector field on the bundle space of the principal bundle.

Theorem Let $\pi: P \rightarrow M$ be a principal G -bundle, if $H: P \rightarrow TP$ is a principal connection on P then there is a unique smooth \mathfrak{g} -valued one-form ω on P such that

- (1) for each $u \in P$, $\text{Ker } \omega_u = H(u) \subseteq T_u P$,
- (2) for each $u \in P$, $\omega_u(\delta_A(u)) = A$, for all $A \in \mathfrak{g}$
- (3) $rg^* \omega = \text{Ad}(g^{-1}) \omega$ for each $g \in G$.

Conversely, given a one-form $\omega: TP \rightarrow \mathfrak{g}$ subject to the conditions (1)-(3) it follows that the function $H: P \rightarrow TP$ defined by $H(u) = \text{Ker } \omega_u$ is a principal connection on P .

Before getting into the proof we first state and prove a useful lemma.

Lemma If G is a Lie group which acts smoothly on the right of a manifold P then

$$d_{rg}(\delta_A(u)) = \delta_{\text{Ad}(g^{-1})A}(ug)$$

for $u \in P$, $g \in G$, $A \in \mathfrak{g}$.

Proof of the Lemma. We first show that for $g \in G$, $A \in \mathfrak{g}$

$$\left. \frac{d}{dt} [g^{-1} \exp(tA) g] \right|_{t=0} = \left. \frac{d}{dt} [\exp(t \operatorname{Ad}(g^{-1})A)] \right|_{t=0}$$

Recall that $\operatorname{Ad}(g): \mathfrak{g} \rightarrow \mathfrak{g}$ is defined by

$$\operatorname{Ad}(g)(A) = \left. \frac{d}{dt} [g \exp(tA) g^{-1}] \right|_{t=0}.$$

Let γ, μ be the curves in G defined by $\gamma(t) = g^{-1} \exp(tA) g$, $\mu(t) = \exp(t \operatorname{Ad}(g^{-1})A)$ then

$$\mu'(t) = \left. \frac{d}{dt} \exp(t \operatorname{Ad}(g^{-1})A) \right|_{t=0} = \operatorname{Ad}(g^{-1})A(\mu(t))$$

since in general $t \rightarrow \exp(tB)$ is the flow of $B \in \mathfrak{g}$ when B is regarded as a left invariant vector field on G . Thus $\mu'(0) = \operatorname{Ad}(g^{-1})A(\mu(0)) = \operatorname{Ad}(g^{-1})A = \gamma'(0)$ and the formula above follows. It now follows that for $u \in \mathfrak{F}$

$$\begin{aligned} d_{rg}(\delta_A(u)) &= d_u \left(\left. \frac{d}{dt} [u \exp(tA)] \right|_{t=0} \right) \\ &= \left. \frac{d}{dt} (r_g [u \exp(tA)]) \right|_{t=0} \\ &= \left. \frac{d}{dt} [(ug) \cdot g^{-1} \exp(tA) g] \right|_{t=0} \\ &= \left. \frac{d}{dt} [(ug) \cdot \exp(t \operatorname{Ad}(g^{-1})A)] \right|_{t=0} \\ &= \delta_{\operatorname{Ad}(g^{-1})A}(ug). \end{aligned}$$

The lemma follows.

Proof of the Theorem. Let H be a principal connection on P . We define a one-form ω on P with values in \mathfrak{G} . First recall that the function $\delta_u: A \rightarrow \mathfrak{S}_A(u)$ is linear for each $u \in P$. We claim that it is a vector space isomorphism and consequently that $V(T_u P)$ may be identified with \mathfrak{G} for each $u \in P$. Using this fact we will be able to modify the Ehresmann connection γ defined by H so as to obtain a \mathfrak{G} -valued one-form ω . Since G acts freely on P it follows from Theorem — that the mapping δ_u is injective. Moreover we know that for each $u \in P$ the fiber $\pi^{-1}(\pi(u))$ of P through u is diffeomorphic to G ; thus

$$\begin{aligned} \dim(V(T_u P)) &= \dim T_u(\pi^{-1}(\pi(u))) \\ &= \dim \pi^{-1}(\pi(u)) \\ &= \dim G = \dim \mathfrak{G}. \end{aligned}$$

Since a linear injection from one vector space to another of the same dimension is necessarily surjective we see that δ_u is a vector space isomorphism. Let $\omega_u = \delta_u^{-1} \circ \gamma_u$ for each $u \in P$. Clearly $\ker \omega_u = \ker \gamma_u = H(u)$ for each $u \in P$. Since γ_u is a projection of $T_u P$ onto $V(T_u P)$ we see that $\gamma_u|_{V(T_u P)}$ is the identity and so $\gamma_u(\delta_u(A)) = \delta_u(A)$ for each $A \in \mathfrak{G}$. Thus for $A \in \mathfrak{G}$,

$$\omega_u(\delta_u(A)) = \delta_u^{-1}(\gamma_u(\delta_u(A))) = \delta_u^{-1}(\delta_u(A)) = A$$

Finally since H is a principal connection, $d_{u,g}(H(u)) = H(ug)$ for all $u \in P$, $g \in G$; thus

$$\begin{aligned} (\tau_g^* \omega)_u(H(u)) &= \omega_{ug}(d_{u,g}(H(u))) \\ &= \omega_{ug}(H(ug)) = \delta_{ug}^{-1}(\gamma_{ug}(H(ug))) = 0 \\ &= \text{Ad}(g^{-1})\omega_u(H(u)). \end{aligned}$$

$$\begin{aligned} \text{Also } (\tau_g^* \omega)_u(\delta_A(u)) &= \omega_{ug}(d_{u,g}(\delta_A(u))) \\ &= \omega_{ug}(\delta_{\text{Ad}(g^{-1})A}(ug)) \\ &= \text{Ad}(g^{-1})A \\ &= \text{Ad}(g^{-1})\omega_u(\delta_A(u)). \end{aligned}$$

It follows that $(\tau_g^* \omega)_u$ and $\text{Ad}(g^{-1})\omega_u$ agree on both $H(u)$ and on $V(T_u P)$ and thus on $T_u P = H(u) \oplus V(T_u P)$. Thus ω satisfies the properties (1)-(3).

Conversely, assume $\omega: TP \rightarrow \mathfrak{g}$ is a 1-form with properties (1)-(3). Define $H: P \rightarrow TP$ by $H(u) = \ker \omega_u \subseteq T_u P$. We show that H is a connection on P . We first show that $T_u P = H(u) \oplus V(T_u P)$. Let $w \in T_u P$ and let $w_v = \delta_{\omega_u(w)}(u) \in V(T_u P)$. We show that the vector $w_h = w - w_v$ is in $H(u)$. We have

$$\begin{aligned} \omega_u(w_h) &= \omega_u(w) - \omega_u(w_v) \\ &= \omega_u(w) - \omega_u(\delta_{\omega_u(w)}(u)) \\ &= \omega_u(w) - \omega_u(w) = 0 \end{aligned}$$

and consequently $w_h \in H(u)$. Thus $T_u P = H(u) + V(T_u P)$

We claim that $H(u) \cap V(T_u P) = \{0\}$. If $w \in H(u) \cap V(T_u P)$ then $w = \delta_A(u)$ for some $A \in \mathfrak{g}$ since the mapping $\delta_u: \mathfrak{g} \rightarrow V(T_u P)$ is an isomorphism from \mathfrak{g} onto $V(T_u P)$ for each $u \in P$. Thus $0 = \omega_u(w) = \omega_u(\delta_A(u)) = A$ and $w = \delta_A(u) = 0$.

It follows that $T_u P = H(u) \oplus V(T_u P)$. We now show that for each vector field X on P , hX and vX are smooth. Since $(vX)(u) = \delta_{w(X(u))}(u)$, it is smooth since it is the composite of smooth mappings. Moreover hX is smooth since $hX = X - vX$. Finally we must show that $d_{u_g}(H(u)) = H(u_g)$ for $u \in P$, $g \in G$. We know that $\omega_{ug} \circ d_{u_g} = \tau_g^* \omega_u = \text{Ad}(g^{-1}) \omega_u$ and consequently

$$w \in H(u) \iff w \in \ker \omega_u \iff \text{Ad}(g^{-1})(\omega_u(w)) = 0$$

$$\iff \omega_{ug}(d_{u_g}(w)) = 0$$

$$\iff d_{u_g}(w) \in H(u_g).$$

Thus $d_{u_g}(H(u)) = H(u_g)$ and the Theorem follows.

In the physics literature connections are usually described as G -valued 1-forms A defined on the underlying space-time manifold relevant to the physical theory, whereas these same connections are described in the mathematical literature as G -valued 1-forms on some principal bundle whose base manifold is the space-time manifold M . To relate the two ways of describing connections one chooses a section (local) S of the principal bundle and then requires that $A = S^* \omega$. The problem with this procedure is that if one changes local sections from S to T then $S^* \omega$ and $T^* \omega$ are not identically G -valued 1-forms on M . This is not a serious problem however as in the physics literature two G -valued 1-forms A and A' are regarded as being equivalent iff they are "gauge equivalent". This means that there must exist a function $g: M \rightarrow G$ such that

$$A' = \bar{g}^{-1} A g + \bar{g}^{-1} dg$$

where we have assumed for simplicity that $G \subseteq \text{GL}(n)$ is a matrix group and $\mathfrak{g} \subseteq \mathfrak{gl}(n)$ its matrix Lie algebra. Thus $\bar{g}^{-1} A g$, $\bar{g}^{-1} dg$ denote matrix operations and the above identity is usually written in components as

$$A'_\mu = \bar{g}^{-1} A_\mu g + \bar{g}^{-1} (\partial_\mu g).$$

in case $g(x) = X(x)I$ for some ^{positive} real-valued function X and $I \in \text{Hom}(V, V)$ the identity matrix one obtains the equation

$$\bar{A}_\mu = A_\mu + X^{-1} \partial_\mu X = A_\mu + \partial_\mu (\ln X).$$

In this form we see that the physicist's A is a generalisation of the notion of a vector potential for an electromagnetic field.

The notion of a connection as we have described it is clearly a geometrical idea which was developed by geometers in the thirties. At roughly the same time Dirac was developing parallel ideas in a totally different language in his description of magnetic monopoles. In the fifties Yang discovered extensions of Dirac's work and early results of Weyl and subsequently initiated modern gauge theory. It turns out that the gauge theories of physics and the theory of connections are equivalent theories as we have briefly indicated above.

To be a little more precise recall that physicists regard A to be gauge equivalent to \bar{A} iff $\bar{A}_\mu = g^{-1} A_\mu g + g^{-1} \partial_\mu g$ for some function $g: M \rightarrow G \subseteq \text{GL}(n)$. On the other hand A and \bar{A} define the same connection on some n -dimensional bundle over

M iff $A = \bar{S}^* \omega$ for one local section \bar{S} of the bundle and $A = S^* \omega$ for another local section S of the bundle. On the other hand if \bar{S}, S are local sections of the same principal fiber bundle we will show below that there exists $g: M \rightarrow G \subseteq GL(n)$ such that $\bar{S}(x) = S(x)g(x)$ for all x . It turns out that if ω is a connection and $g: M \rightarrow G$ is a mapping which relates the sections \bar{S} and S as above then

$$\bar{S}^* \omega = g^{-1} S^* \omega g + g^{-1} dg!$$

In other words the relationship between $\bar{S}^* \omega$ and $S^* \omega$ is precisely what is required to be compatible with the physicists' notion of gauge equivalence.

Since it is the equivalence class of gauge fields which is of physical interest rather than one particular element A of the class, this tells us that mathematically the emphasis should be on connections. Our description is a bit oversimplified however as not every principal bundle admits global sections. On the other hand every principal bundle over Minkowski space does admit global sections and so our account is accurate when the space-time M is Minkowski space and the gauge group G is a subgroup of $GL(n)$ for some n .

Theorem Assume $\pi: P \rightarrow M$ is a principal G -bundle and that ω is a connection on P . Let $\sigma: U \rightarrow P$, $\bar{\sigma}: \bar{U} \rightarrow P$ denote local sections of π such that $U \cap \bar{U} \neq \emptyset$, and $g: U \cap \bar{U} \rightarrow G$ is the unique mapping such that

$$\bar{\sigma}(x) = \sigma(x) \cdot g(x)$$

for all $x \in U \cap \bar{U}$, then

$$(\bar{\sigma}^* \omega)_x = \text{Ad}(g(x)^{-1}) (\sigma^* \omega)_x + d_x \ell_{g(x)}^{-1} \circ dg_x$$

for each x .

Proof. Note that $\bar{\sigma}(x) = r_{g(x)}(\sigma(x))$ for $x \in U \cap \bar{U}$ by hypothesis. We show first that for $x \in M$, $v \in T_x M$

$$d_x \bar{\sigma}(v) = d_x r_{g(x)}(d_x \sigma(v)) + \mathcal{A}_A(\bar{\sigma}(x))$$

where $A = d_x \ell_{g(x)}^{-1}(dg_x(v)) \in T_e G = \mathfrak{g}$. To see this let $\gamma: (-a, a) \rightarrow M$ denote a curve in M such that $\gamma(0) = x$ and $\gamma'(0) = v$. Then $\bar{\sigma}(\gamma(t)) = r_{g(\gamma(t))}(\sigma(\gamma(t)))$ for all t and

$$d_x \bar{\sigma}(v) = \frac{d}{dt} [\bar{\sigma}(\gamma(t))] \Big|_{t=0} = \frac{d}{dt} [r_{g(\gamma(t))}(\sigma(\gamma(t)))] \Big|_t$$

Let $f: (-a, a) \times (-a, a) \rightarrow M$ be defined by

$$f(t_1, t_2) = r_{g(\gamma(t_1))}(\sigma(\gamma(t_2)))$$

$$\left. \frac{d}{dt} [r_{g(x(t))}(\alpha(x(t)))] \right|_{t=0} = (\partial_1 f)(0,0) + (\partial_2 f)(0,0)$$

But

$$(\partial_2 f)(t_1, t_2) = dr_{g(x(t_1))} \left(\frac{d\alpha}{dt}(x'(t_2)) \right)$$

and

$$(\partial_2 f)(0,0) = dr_{g(x)} \left(\frac{d\alpha}{dt}(v) \right).$$

To compute $(\partial_1 f)(0,0)$ first let $\sigma_u: G \rightarrow P$ be defined by $\sigma_u(a) = u \cdot a$ for $u \in P, a \in G$.

Then

$$\partial_1 f(t,0) = \frac{d}{dt} [\alpha(x(t)) \cdot g(x(t))]$$

$$= \frac{d}{dt} [\alpha(x) \cdot (g(x) g(x)^{-1} g(x(t)))]$$

$$= \frac{d}{dt} [\sigma_{\alpha(x)} (l_{g(x)^{-1}}(g(x(t))))]$$

$$= d\sigma_{\alpha(x)} (dl_{g(x)^{-1}}(dg(x'(t))))$$

for each $t \in (-a, a)$. It follows that

$$(\partial_1 f)(0,0) = d\sigma_{\alpha(x)} (dl_{g(x)^{-1}}(dg_x(v))).$$

Since $dl_{g(x)^{-1}}(dg_x(v)) \in T_e G$ we can identify it with an element A of the Lie algebra \mathfrak{g} of G and consequently

$$(\partial_1 f)(0,0) = d\sigma_{\alpha(x)}(A) = \int_A (\pi(x)).$$

Thus

$$d\pi_x(v) = dr_{g(x)} \left(\frac{d\alpha}{dt}(v) \right) + \int_A (\pi(x))$$

as we asserted at the beginning of the proof.

It now follows that

$$\begin{aligned}
 \omega(d\pi(v)) &= \omega\left(\frac{dr}{dr} g(x) \left(\frac{ds}{ds}(v)\right)\right) + \omega_{\pi(x)}\left(\int_A(\pi(x))\right) \\
 &= (r_{g(x)}^* \omega)\left(\frac{ds}{ds}(v)\right) + A \\
 &= \text{Ad}(g(x)^{-1}) \omega\left(\frac{ds}{ds}(v)\right) + A \\
 &= \text{Ad}(g(x)^{-1}) (\pi^* \omega)_x(v) + A
 \end{aligned}$$

and

$$(\pi^* \omega)_x(v) = \text{Ad}(g(x)^{-1}) (\pi^* \omega)_{\pi(x)}(v) + d\ell_{g(x)^{-1}}(dg(v))$$

The theorem follows.

Corollary Let $\pi: P \rightarrow M$ be a principal G -bundle and ω a connection on P . If $\{s_\alpha\}$ is a family of local sections of π , $s_\alpha: U_\alpha \rightarrow \pi^{-1}(U_\alpha)$ such that $\{U_\alpha\}_{\alpha \in I}$ covers M and if $A_\alpha = s_\alpha^* \omega$ for each α , then

$$A_\beta(x) = \text{Ad}\left(g_{\alpha\beta}(x)^{-1}\right) A_\alpha(x) + \left(d\ell_{g_{\alpha\beta}(x)^{-1}} \circ dg_{\alpha\beta}\right)$$

where

$$s_\beta(x) = s_\alpha(x) g_{\alpha\beta}(x)$$

for $x \in U_\alpha \cap U_\beta$.

This corollary is an immediate consequence of the theorem.