

There are four fields which are fundamental in electromagnetic theory, denoted $\vec{E}, \vec{D}, \vec{H}, \vec{B}$.
 Each of these is defined on Minkowski space M . Minkowski space is a manifold with a distinguished atlas A . Each chart belonging to A has domain all of M and so is a bijection from M onto \mathbb{R}^4 . If x, y are both in A , then $y \circ x^{-1}$ is a Lorentz transformation of \mathbb{R}^4 . This means

that $y \circ x^{-1}: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is linear and that it preserves the the Lorentz "inner-product" defined on \mathbb{R}^4 by

$$\langle v, w \rangle = v^0 w^0 - v^1 w^1 - v^2 w^2 - v^3 w^3.$$

Thus if $\Lambda = y \circ x^{-1}$ one has that

$$\langle \Lambda(v), \Lambda(w) \rangle = \langle v, w \rangle$$

for all $v, w \in \mathbb{R}^4$. One may define a metric η on M by

$$\eta\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = \eta_{\mu\nu} \quad \mu, \nu = 0, 1, 2, 3$$

where

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The mapping from $T_p M$ to \mathbb{R}^4 defined 42
 by $\lambda^\mu(\frac{\partial}{\partial x^\mu}|_p) \rightarrow (\lambda^0, \lambda^1, \lambda^2, \lambda^3)$

is then an isometry:

$$\eta(\lambda^\mu \frac{\partial}{\partial x^\mu}, \rho^\nu \frac{\partial}{\partial x^\nu}) = \langle (\lambda^0, \lambda^1, \lambda^2, \lambda^3), (\rho^0, \rho^1, \rho^2, \rho^3) \rangle.$$

Consequently we often identify (M, η) with $(\mathbb{R}^4, \langle \cdot, \cdot \rangle)$. Elements of A are called inertial coordinates, inertial frames, or inertial observers. The term "coordinate" is better as inertial frame ought to refer to $(\frac{\partial}{\partial x^\mu})$.

Fix a chart $x: M \rightarrow \mathbb{R}^4$ in A , an inertial coordinate system. Write

$$(x^\mu) = (x^0, x^1, x^2, x^3).$$

We write $x^0 = ct$ where c is the speed of light and choose units so that $c=1$.

The fields \vec{E} , \vec{D} , \vec{H} , \vec{B} are functions of $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ but are vectors in \mathbb{R}^3 . For example

$$\vec{E}(t, x, y, z) \in T_{(x,y,z)} \mathbb{R}^3.$$

with similar conventions for $\vec{D}, \vec{H}, \vec{B}$

Here we have identified M with \mathbb{R}^4 43
 and $\vec{E}, \vec{D}, \vec{H}, \vec{B}$ are "time varying"
 vector fields on \mathbb{R}^3 relative to our fixed
 inertial frame.

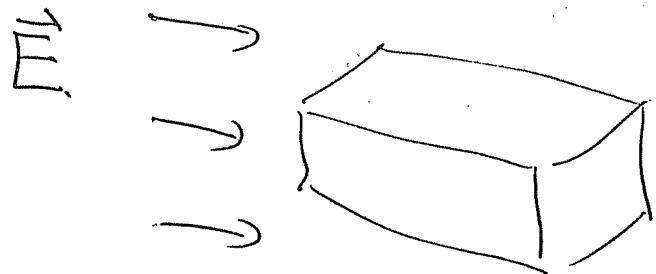
In elementary electromagnetism \vec{E}
 is called the electric field measured
 in volts/meter. Generally there is a
 function $\varphi: \mathbb{R}^4 \rightarrow \mathbb{R}$ called the scalar
 potential such that $\vec{E} = -\nabla \varphi$,

$$\vec{E}(t, x, y, z) = -\frac{\partial \varphi}{\partial x} \hat{i} - \frac{\partial \varphi}{\partial y} \hat{j} - \frac{\partial \varphi}{\partial z} \hat{k}.$$

If C is a curve from an equipotential
 surface $\{(t, x, y, z) \mid \varphi(t, x, y, z) = R_1\}$
 to another such surface $\{(t, x, y, z) \mid \varphi(t, x, y, z) = R_2\}$
 then $\int_C \vec{E} \cdot d\vec{s}$ is the difference in the
 potentials, $R_2 - R_1$ and is measured in volts.
 Since one-forms can be integrated along curves
 this suggests that $E = E_1 dx + E_2 dy + E_3 dz$
 $= E_i dx$ ($i=1, 2, 3$) is the differential
 form analogue of \vec{E} . Moreover physics
 tell us \vec{E} is a polar vector, meaning
 that \vec{E} changes signs relative to the coordinate

Change $(x^1, x^2, x^3) \rightarrow (-x^1, -x^2, -x^3)$. This is compatible with the choice of \vec{E} as a 1-form but would not be compatible if one tried to model \vec{E} as a 2-form.

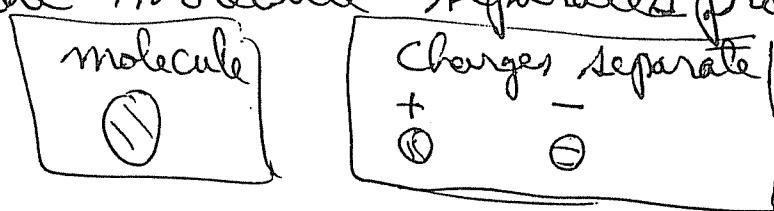
The field \vec{D} is called the displacement field, it is useful when an electric field is imposed in the presence of matter.



A molecule of matter will react to an electric field by experiencing a separation

of its positive and negative charged components

Thus the molecule separates producing a dipole



And a corresponding field \vec{P} called the polarization vector. \vec{D} is then defined

by $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

where ϵ_0 is a constant called the permittivity of the matter on which \vec{E} is imposed.

Now \vec{D} is measured in Coulombs/meter². It measures flux through a "closed" surface

S and $\int_S \vec{D} \cdot d\vec{A}$ is the "free charge" 45
 contained in S (free charge consists of charged
 particles not bound to other molecules of matter).

Thus \vec{E} is associated with total charge, and
 \vec{D} with free charge. \vec{E} is the negative
 of a gradient but \vec{D} is not conservative,
 it is not curl free in general. Finally
 \vec{D} is an axial vector, it is invariant
 under transformation $(x^1, x^2, x^3) \rightarrow (-x^1, -x^2, -x^3)$.
 All these properties suggest that the
 differential form analogue of \vec{D} is a
 two form:

$$D = \pm \epsilon_{ijk} D^i (dx^j \wedge dx^k)$$

$$D = D^1 (dy \wedge dz) + D^2 (dz \wedge dx) + D^3 (dx \wedge dy).$$

Now consider magnetic fields. Assume
 that \vec{B}_0 is a magnetic field imposed in the
 presence of matter. There is a field called
magnetization vector \vec{M} which plays a role
 similar to that of \vec{P} as it relates to \vec{E} .

There are magnetic dipole moments induced in matter in the presence of \vec{B}_0 and \vec{M} which accounts for these moments. The total field $\vec{B} = \vec{B}_0 + \mu_0 \vec{M}$ is called the magnetic induction field where μ_0 is an appropriate constant. If C is a curve which is the boundary of some surface S , then $I_M = \oint_C \vec{M} \cdot d\vec{\sigma}$ is defined to be the magnetization current through S . It turns out that if I is the real conduction current then $\oint_C \vec{B}_0 \cdot d\vec{\sigma} = \mu_0 I$, so that

$$\oint_C \vec{B} \cdot d\vec{\sigma} = \oint_C (\vec{B}_0 + \mu_0 \vec{M}) \cdot d\vec{\sigma} = \mu_0 (I + I_M)$$

It follows that the real conduction current I is given by

$$I = \oint_C \left(\frac{\vec{B} - \mu_0 \vec{M}}{\mu_0} \right) \cdot d\vec{\sigma}.$$

and $\vec{H} = \frac{\vec{B} - \mu_0 \vec{M}}{\mu_0}$ defined by this equation is called the magnetic field strength.

Maxwell's equations encode all the information regarding electrostatics, magnetostatics, and the dynamical features of interacting magnetic and electric fields. They are

$$\boxed{\text{Div } \vec{B} = 0 \quad \text{Curl } \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0}$$

$$\boxed{\text{Div } \vec{D} = \rho \quad \text{Curl } \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}}$$

Where ρ is charge density and \vec{j} is the conduction current defined throughout M . These equations are not enough however as one must also have equations relating to the interaction of the fields with matter. These are called Constitutive equations and for a large class of interactions are given by

$$\boxed{\vec{B} = \mu \vec{H} \quad \vec{D} = \epsilon \vec{E}}$$

for appropriate constants μ, ϵ . Note that without the constitutive equations the dynamical equations given above are decoupled as two of them relate

48

to \vec{B} and \vec{E} while the other two relate to \vec{D} and \vec{H} . This decoupling really never occurs however as the constants μ, ϵ are not zero even in vacuum electromagnetism.

We now wish to rewrite these basic equations in the language of differential forms. Using arguments similar to those used for \vec{E} and \vec{D} we see that it is natural to model \vec{H} as a 1-form and \vec{B} as a 2-form. Indeed \vec{H} was defined so that its line integral gives the conduction current through a surface:

$$\int_C \vec{H} \cdot d\vec{\sigma} = I$$

where the curve C is a boundary of the surface. This suggests that \vec{H} should be modelled as a 1-form. The fact that \vec{H} is a polar vector contributes to the conclusion as 1-forms on \mathbb{R}^3 change signs under the transformation $(x^1, x^2, x^3) \rightarrow (-x^1, -x^2, -x^3)$. The case for \vec{B} is a little more difficult to argue but it is an axial vector and 2-forms are invariant under the transformation above and so are axial in nature.

From the considerations indicated above we

write $E = E_i dx^i = E_1 dx + E_2 dy + E_3 dz$

$$H = H_i dx^i = H_1 dx + H_2 dy + H_3 dz$$

$$D = \frac{1}{2} \varepsilon_{ijk} H^i (dx^j \wedge dx^k) = H^1(dy \wedge dz) + H^2(dz \wedge dx) + H^3(dx \wedge dy)$$

$$B = \frac{1}{2} \varepsilon_{ijk} B^i (dx^j \wedge dx^k) = B^1(dy \wedge dz) + B^2(dz \wedge dx) + B^3(dx \wedge dy)$$

Notice that the components of D, B are the same as those of the corresponding vector fields

$$\vec{D} = D^1 \frac{\partial}{\partial x} + D^2 \frac{\partial}{\partial y} + D^3 \frac{\partial}{\partial z}$$

$$\vec{B} = B^1 \frac{\partial}{\partial x} + B^2 \frac{\partial}{\partial y} + B^3 \frac{\partial}{\partial z}$$

but the covariant form of \vec{E}, \vec{H} requires

$$E_i = \eta_{ij} E^j = -E^i, \quad H_i = \eta_{ij} H^j = -H^i.$$

Since all the fields $\vec{H}, \vec{D}, \vec{E}, \vec{B}$ are "time varying" vector fields on \mathbb{R}^3 and the corresponding forms are to be defined on Minkowski space which we are identifying with \mathbb{R}^4 we need to compare the exterior derivatives d^4 on \mathbb{R}^4 with d^3 on \mathbb{R}^3 .

So assume \vec{X}, \vec{Y} are arbitrary "time-varying" vector fields on \mathbb{R}^3 . We consider the 1-form \vec{X} defined by \vec{X} and the 2-form \vec{Y} defined

by \vec{Y} :

$$\vec{X} = X_i dx^i = (-\vec{X}^1)dx + (-\vec{X}^2)dy + (-\vec{X}^3)dz.$$

$$\vec{Y} = \frac{1}{2} \epsilon_{ijk} Y^j (dx \wedge dy \wedge dz) = Y^1(dy \wedge dz) + Y^2(dz \wedge dx) + Y^3(dx \wedge dy).$$

These may be regarded as forms on both \mathbb{R}^3 and \mathbb{R}^4 because of the time dependence of the functions X^i , Y^j . By a previous exercise

$$d^3 \vec{X} = -(\text{curl } \vec{X})^1(dy \wedge dz) - (\text{curl } \vec{X})^2(dz \wedge dx) - (\text{curl } \vec{X})^3(dx \wedge dy)$$

and $d^3 \vec{Y} = (\text{Div } \vec{Y})(dx \wedge dy \wedge dz)$. The operator d^4 must take the time coordinate t into account.

If we denote $-\frac{\partial \vec{X}}{\partial t} dx - \frac{\partial \vec{X}}{\partial t} dy - \frac{\partial \vec{X}}{\partial t} dz$ by $\frac{\partial \vec{X}}{\partial t}$ with a similar notation for \vec{Y} we see

that

$$d^4 \vec{X} = (dt \wedge \frac{\partial \vec{X}}{\partial t}) + d^3 \vec{X}$$

$$d^4 \vec{Y} = (dt \wedge \frac{\partial \vec{Y}}{\partial t}) + d^3 \vec{Y}.$$

Applying these considerations to $\vec{E}, \vec{D}, \vec{H}, \vec{B}$ we have

$$d^4 E = (dt \wedge \frac{\partial E}{\partial t}) + d^3 E, \quad d^4 H = (dt \wedge \frac{\partial H}{\partial t}) + d^3 H$$

$$d^4 B = (dt \wedge \frac{\partial B}{\partial t}) + (\text{Div } \vec{B})(dx \wedge dy \wedge dz)$$

$$d^4 D = (dt \wedge \frac{\partial D}{\partial t}) + (\text{Div } \vec{D})(dx \wedge dy \wedge dz).$$

At this point we transform Maxwell's equations into differential form notation.

Recall that the time evolution equations are

$$\text{Curl} \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \text{Div} \vec{B} = 0$$

$$\text{Curl} \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j} \quad \text{Div} \vec{D} = \rho$$

where $\vec{j} = j^1 \frac{\partial}{\partial x} + j^2 \frac{\partial}{\partial y} + j^3 \frac{\partial}{\partial z}$ is a current vector and ρ is charge density. We also have constitutive equations $\vec{B} = \mu \vec{H}$, $\vec{D} = \epsilon \vec{E}$.

$$\text{Consider the equation } \text{Curl} \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}$$

In differential form notation this becomes

$$d^3(-H) - \frac{\partial D}{\partial t} = j^1 (dy \wedge dz) + j^2 (dz \wedge dx) + j^3 (dx \wedge dy)$$

or

$$\frac{\partial D}{\partial t} - d^3(-H) = -j = -j^1 (dy \wedge dz) - j^2 (dz \wedge dx) - j^3 (dx \wedge dy)$$

But

$$d^4(D - dt \wedge H) = d^4 D - (-1)(dt \wedge d^4 H)$$

$$= d^4 D - dt \wedge d^3(-H)$$

$$= \left[(dt \wedge \frac{\partial D}{\partial t}) + \text{Div} \vec{D} (dx \wedge dy \wedge dz) \right] - dt \wedge d^3(-H)$$

$$= dt \wedge \left(\frac{\partial D}{\partial t} - d^3(-H) \right) + \text{Div} \vec{D} (dx \wedge dy \wedge dz)$$

$$= \rho (dx \wedge dy \wedge dz) - j^1 (dt \wedge dy \wedge dz)$$

$$- j^2 (dt \wedge dz \wedge dx) - j^3 (dt \wedge dx \wedge dy)$$

We denote this last form by J and refer to it as the current 3-form although its

$dx \wedge dy \wedge dz$ component p is charge density.

Let $G = D - (dt \wedge H)$, then two of Maxwell's equations become

$$dG = J$$

although we only proved that the two equations imply that $dG = J$. The converse is left to the reader. Now the remaining evolution equations are $\text{Curl } \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$, $\text{Div } \vec{B} = 0$

As before, the first of these becomes

$$d^3(-E) + \frac{\partial \vec{B}}{\partial t} = 0 \text{ and so}$$

$$\begin{aligned} d^4(B + dt \wedge E) &= d^4B + (-1)(dt \wedge d^4E) \\ &= (dt \wedge \frac{\partial \vec{B}}{\partial t}) + \text{Div } \vec{B}(dx \wedge dy \wedge dz) \\ &\quad + (dt \wedge d^3(-E)) \\ &= dt \wedge \left[\frac{\partial \vec{B}}{\partial t} + d^3(-E) \right] + (\text{Div } \vec{B})(dx \wedge dy \wedge dz) \\ &= 0 \end{aligned}$$

So if $F = B + dt \wedge E$ the other two evolution equations are $dF = 0$.

Now the constitutive equations $\vec{B} = \mu \vec{H}$, $\vec{D} = \epsilon \vec{E}$ are somewhat inconsistent in the usual vector form as both of them imply that a constant multiple of polar vector is an axial vector. In differential

In form language this is easily resolved by requiring that $B = -\mu *^3 H$, $D = -\epsilon *^3 E$.

This converts the 1-forms H, E to 2-forms B, D as required. With these choices we will show below that for appropriate μ, ϵ

$$*^4 F = \mu G$$

This completes the transformation of Maxwell's equations to form notation. In order to prove this last identity we need to obtain relations comparing the Hodge dual $*^4$ on \mathbb{R}^4 to the Hodge dual $*^3$ on \mathbb{R}^3 since our forms E, B, H, D may be regarded both as forms on \mathbb{R}^4 and as "time-varying" forms on \mathbb{R}^3 .

In vector field notation the constitutive relations are $\vec{B} = \mu \vec{H}$, $\vec{D} = \epsilon \vec{E}$ for constants μ, ϵ .

It is known that for electromagnetic waves to propagate at light speed one must have

$$\epsilon \mu = \frac{1}{c^2}$$

where c is light speed. We have chosen units so that $c=1$ so we may assume that $\epsilon \mu = 1$.

Now $\vec{B} = \mu \vec{H}$ implies and is implied by

$B^i = \mu H^i$ for all $i=1, 2, 3$. These equations are equivalent to $*^3 B = *^3 (\vec{B}^1(dx \wedge dz) + \vec{B}^2(dx \wedge dy) + \vec{B}^3(dy \wedge dz)) = \mu *^3 (\vec{H}^1(dx \wedge dz) + \vec{H}^2(dx \wedge dy) + \vec{H}^3(dy \wedge dz)) =$

$$= -\mu (H_1 dx + H_2 dy + H_3 dz) = -\mu H \quad 54$$

Similarly $\vec{D} = \epsilon \vec{E}$ is equivalent to $*^3 D = -\epsilon E$

In Minkowski space $M \cong \mathbb{R}^4$

$$*^4(dx^2 \wedge dx^3) = dx^0 \wedge dx^1$$

$$*^4(dx^3 \wedge dx^1) = dx^0 \wedge dx^2$$

$$*^4(dx^1 \wedge dx^2) = dx^0 \wedge dx^3$$

$$*^4(dx^0 \wedge dx^1) = dx^3 \wedge dx^2$$

$$*^4(dx^0 \wedge dx^2) = dx^1 \wedge dx^3$$

$$*^4(dx^0 \wedge dx^3) = dx^2 \wedge dx^1$$

While in Euclidean 3-space \mathbb{R}^3 ,

$$*^3 dx^1 = dx^2 \wedge dx^3$$

$$*^3 dx^2 = dx^3 \wedge dx^1$$

$$*^3 dx^3 = dx^1 \wedge dx^2.$$

Now E may be regarded as a time-varying 1-form on \mathbb{R}^3 and it may also be regarded as a 1-form on \mathbb{R}^4 . Thus $E = E_1 dx + E_2 dy + E_3 dz$ gives

$$*^3 E = E_1 (dx^2 \wedge dx^3) + E_2 (dx^3 \wedge dx^1) + E_3 (dx^1 \wedge dx^2)$$

$$= E_1 *^4(dx^3 \wedge dx^0) + E_2 *^4(dx^2 \wedge dx^0) + E_3 *^4(dx^1 \wedge dx^0)$$

$$= *^4(E \wedge dt) = -*^4(dt \wedge E)$$

and $B = B^1(dy \wedge dz) + B^2(dz \wedge dx) + B^3(dx \wedge dy)$ gives

$$dt \wedge *^3 B = B^1(dt \wedge dx) + B^2(dt \wedge dy) + B^3(dt \wedge dz)$$

$$= B^1 *^4(dx^2 \wedge dx^3) + B^2 *^4(dx^3 \wedge dx^1) + B^3 *^4(dx^1 \wedge dx^2)$$

$$= *^4[B^1(dy \wedge dz) + B^2(dz \wedge dx) + B^3(dx \wedge dy)] = *^4 B.$$

Note that we have used the fact that $E^i = \eta^{ij} E_j$
 $= -E_i$, but we have not needed to transform
the components B^i as $B = \frac{1}{2} \epsilon_{ijk} B^i (dx^j \wedge dx^k)$
is defined directly in terms of the components
 B^i of \vec{B} . Finally,

$$\begin{aligned} *^4 F &= *^4(B + dt \wedge E) \\ &= *^4 B + *^4(dt \wedge E) \\ &= (dt \wedge (*^3 B)) + (-1)(*^3 E) \\ &= (-\mu)(dt \wedge H) + (-1)(-\frac{1}{\epsilon}) D \quad (*^3 D = -\epsilon (*^3 E) \\ &= \mu(D - dt \wedge H) \quad (\mu = \frac{1}{\epsilon}) \\ &= \mu G. \end{aligned}$$

So Maxwell's equations become

$$dF = 0, \quad dG = J, \quad *^4 F = \mu G.$$

They can also be written as

$$dF = 0, \quad d(*^4 F) = \mu J$$

Let $j = *^4(\mu J)$ then $*^4 d(*^4 F) = j$

and since F is a 2-form in Minkowski space we have $\delta F = *^4 d(*^4 F) = j$

and Maxwell's equations become

$$dF = 0 \quad \delta F = j$$

We will show later that $dF = 0 \Leftrightarrow dA = F$.

Remark 1

Often it is useful have explicit formulae for the matrices of F and G . They are

$$(F_{\mu\nu}) = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{bmatrix}$$

$$(G_{\mu\nu}) = \begin{bmatrix} 0 & H^1 & H^2 & H^3 \\ -H^1 & 0 & D^3 & -D^2 \\ -H^2 & -D^3 & 0 & D^1 \\ -H^3 & D^2 & -D^1 & 0 \end{bmatrix}$$

Remark 2

If $(x^\mu), (\bar{x}^\nu)$ are two sets of internal coordinates on Minkowski space, then there exists a Lorentz matrix $\Lambda = (\Lambda^\alpha_\beta)$ such that $\bar{x}^\mu = \Lambda^\mu_\alpha x^\alpha$

and

$$\begin{aligned} F &= \frac{1}{2} \bar{F}_{\mu\nu} (\bar{dx}^\mu \wedge \bar{dx}^\nu) = \frac{1}{2} \bar{F}_{\mu\nu} (\Lambda^\mu_\alpha \Lambda^\nu_\beta) (dx^\alpha \wedge dx^\beta) \\ &= \frac{1}{2} F_{\alpha\beta} (dx^\alpha \wedge dx^\beta) \end{aligned}$$

where $F_{\alpha\beta} = \bar{F}_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta$ which is the usual transformation law of a tensor. Thus F and similarly G are well-defined tensors and can be expressed in any coordinate chart on M . It follows that $dF = 0$ and

$dG = \mu J$ are well defined tensor identities and these two equations are invariant under

the entire group of diffeomorphisms. On the 57
other hand the equation $*^4F = -\mu G$ depends
on the metric since the Hodge dual does.

We show below that the symmetries of
 $dF = 0, dG = \mu J$ is broken to the
Lorentz group when the equation $*^4F = -\mu G$
is added to the system. In particular

$*^4F = -\mu G$ is not invariant under
Galilean transformations although Newton's
equations are. This fact suggests that
either Maxwell's equations are wrong or
Newton's laws are not universal. It turns
out that the latter is the case. Special
relativity is invariant under Lorentz
transformations and that theory leads to
the correction of Newton's laws. The next
couple of theorems and Exercises
lead to a proof that Maxwell's equations
are invariant relative to Lorentz transformation

Theorem 1.22. Let M be a manifold and g a metric on M . If $\{e_i\}$ is a g -orthonormal basis of $T_p M$ for some $p \in M$ then

$$\{e^{i_1} \wedge \dots \wedge e^{i_R} \mid i_1 < i_2 < \dots < i_R\}$$

is an orthonormal basis of $(\Lambda^k M)_p$ relative to \tilde{g} .

Moreover the index of \tilde{g} is the number of increasing sequences $i_1 < i_2 < \dots < i_k$ for which

$$g(e_{i_1}, e_{i_1}) g(e_{i_2}, e_{i_2}) \dots g(e_{i_k}, e_{i_k}) = -1.$$

Proof Let $\alpha = e^{i_1} \wedge \dots \wedge e^{i_R}$, $\beta = e^{j_1} \wedge \dots \wedge e^{j_R}$

where $i_1 < i_2 < \dots < i_R$ and $j_1 < j_2 < \dots < j_R$.

Then

$$\tilde{g}(\alpha, \beta) = \alpha_{r_1 r_2 \dots r_k} \beta^{r_1 r_2 \dots r_k}$$

and since

$$\alpha = \sum (-1)^{\sigma} (e^{i_{\sigma(1)}} \otimes \dots \otimes e^{i_{\sigma(R)}})$$

we see that if $r_1 < r_2 < \dots < r_k$ then

the only coefficient $\alpha_{r_1 r_2 \dots r_k}$ is when $r_a = i_a, \forall a$
+ Thompson

On the other hand $\beta^{r_1 r_2 \dots r_k} = (g^{r_1 A_1} \dots g^{r_k A_k}) \beta_{A_1 \dots A_k}$

Since $\alpha_{r_1 \dots r_k} = 0$ except when $r_a = i_a$

$$\tilde{g}(\alpha, \beta) = \alpha_{i_1 \dots i_k} \beta^{i_1 \dots i_k} \quad (\text{no sum})$$

$$= \alpha_{i_1 \dots i_k} g^{i_1 A_1} \dots g^{i_k A_k} \beta_{A_1 \dots A_k} \quad (\text{no sum or } \{i_a\})$$

$$= \alpha_{i_1 \dots i_k} g^{i_1 i_1} \dots g^{i_k i_k} \beta_{i_1 \dots i_k} \quad (\text{no sum})$$

$$= g^{i_1 i_1} \dots g^{i_k i_k} \alpha_{i_1 \dots i_k} \beta_{i_1 \dots i_k}$$

$$= g_{i_1 i_1} \dots g_{i_k i_k} \beta_{i_1 \dots i_k}$$

If $r_1 < r_2 < \dots < r_k$ then $\beta_{r_1 \dots r_k}$ is nonzero iff $r_a = j_a$ for all a . Thus for $\alpha = e^{i_1} \wedge \dots \wedge e^{i_k}$, $\beta = e^{j_1} \wedge \dots \wedge e^{j_k}$, $\tilde{g}(\alpha, \beta) = 0$ unless $\alpha = \beta$ in which case $\tilde{g}(\alpha, \alpha) = g_{i_1 i_1} \dots g_{i_k i_k} = \pm 1$. Thus $\{e^{i_1} \wedge \dots \wedge e^{i_k} \mid i_1 < i_2 < \dots < i_k\}$ is an orthonormal basis of $\Lambda^k T_p M$. It is clear also that the index of \tilde{g} is the cardinality of the set of all $i_1 < i_2 < \dots < i_k$ such that $g_{i_1 i_1} g_{i_2 i_2} \dots g_{i_k i_k} = -1$.

Definition 1.23 If V is a vector space then the set of all invertible linear transformation from V onto V is a group under function composition. We denote this group by $GL(V)$. In Exercise 1.7 the reader is invited to show that $GL(V)$ acts on $\Lambda^k V^*$ via

$$(L \cdot \alpha)(v_1, v_2, \dots, v_k) = \alpha(L^{-1}(v_1), L^{-1}(v_2), \dots, L^{-1}(v_k))$$

for all $L \in GL(V)$, $\alpha \in \Lambda^k V^*$, $v_1, v_2, \dots, v_k \in V$.

This means that if $L_1, L_2 \in GL(V)$ and i is the identity mapping on V , then, for all $\alpha \in \Lambda^k V^*$,

$$(L_1 \circ L_2) \cdot \alpha = L_1 \cdot (L_2 \cdot \alpha)$$

$$i \cdot \alpha = \alpha.$$

Notice that $L \cdot \alpha$ is just the pullback of α under L^{-1} .

60

Definition 1.24 If V and W are vector spaces with metrics g and h respectively then a linear mapping $L: V \rightarrow W$ is an isometry iff L is invertible and $h(L(v), L(w)) = g(v, w)$ for all $v, w \in V$.

In case L is an isometry it can be shown that for forms $\alpha, \beta \in \Lambda^p W^*$

$$\tilde{g}(L^* \alpha, L^* \beta) = \tilde{h}(\alpha, \beta).$$

If μ_g, μ_h are volumes on V and W respectively which are metric compatible, we say that L is orientation preserving iff $L^* \mu_h = \mu_g$.

Theorem 1.25 Let V and W be finite dimensional vector spaces with metrics g and h along with compatible volumes μ_g and μ_h , respectively.

If $L: V \rightarrow W$ is an orientation preserving isometry then $*(\bar{L}^*\beta) = L^*(\beta)$ for each $\beta \in \Lambda^k W^*$.

Proof Observe that for $\beta \in \Lambda^k W^*$, $L^*\beta \in \Lambda^{k-R} V^*$ and $*(\bar{L}^*\beta)$, $L^*(\beta) \in \Lambda^{m-k} V^*$ where $m = \dim V$. Moreover, for each $\alpha \in \Lambda^{n-k} V^*$,

$$\begin{aligned}\tilde{g}(\alpha, L^*(\beta)) \mu_g &= \tilde{g}(L^*((L^{-1})^*\alpha), *(\beta)) \mu_g \\ &= \tilde{h}((L^{-1})^*\alpha, *(\beta)) (L^*\mu_g) \\ &= L^*(\tilde{h}((L^{-1})^*\alpha, *(\beta)) \mu_h) \\ &= L^*[(L^{-1})^*\alpha \wedge *(\beta)] \\ &= (-1)^{k(n-k)+k+1} L^* ((L^{-1})^*\alpha \wedge \beta) \\ &= (-1)^{k(n-k)+k+1} (\alpha \wedge (L^*\beta)) \\ &= \alpha \wedge *(\bar{L}^*\beta) \\ &= \tilde{g}(\alpha, *(\bar{L}^*\beta)) \mu_g\end{aligned}$$

Thus $\tilde{g}(\alpha, L^*(\beta) - *(\bar{L}^*\beta)) = 0$ for all α and $L^*(\beta) = *(\bar{L}^*\beta)$.

Remark We used the fact that

$$\tilde{g}(L^*\beta, L^*\gamma) = \tilde{h}(\beta, \gamma)$$

for all β, γ in $\Lambda^\infty W^*$. This is true as it follows from the identity

$$(\star) \quad \tilde{g}(L^{-1}\cdot\beta, L^{-1}\cdot\gamma) = \tilde{h}(\beta, \gamma)$$

and the fact that $L^{-1}\cdot\beta = ((L^{-1})^{-1})^*\beta = L^*\beta$. with a similar identity for γ . The reader is invited to prove (\star) .

If M is a manifold with a metric g

then $\varphi: M \rightarrow M$ is an isometry iff

$d\varphi: T_p M \rightarrow T_{\varphi(p)} M$ is an isometry

from $(T_p M, g_p)$ onto $(T_{\varphi(p)} M, g_{\varphi(p)})$.

It is orientation preserving if

$\varphi^*(\mu_g) = \mu_{\varphi(g)}$ where μ_g is a volume on M compatible with g .

Corollary Assume that M is a manifold, that g is a metric on M and that μ_g is a volume on M which is compatible with g . If $\varphi: M \rightarrow M$ is an orientation preserving isometry then $\varphi^*(\ast\beta) = \ast\varphi^*\beta$ for every $\beta \in \Omega^k M$.

Proof This follows from the theorem if for $p \in M$,

$$V = T_p M, W = T_{\varphi(p)} M \text{ and } L = d_p \varphi.$$

Let $M = \mathbb{R}^4$ and let η be the Minkowski metric on \mathbb{R}^4 .
 Let L^+ denote the group of matrices $A \in GL(\mathbb{R})$
 such that A is an orientation preserving isometry of (M, η) , thus

$$A \eta A^T = \eta \quad \det A = 1.$$

Recall that

$$\eta(A \cdot x, A \cdot y) = \eta(x, y)$$

for all $x, y \in \mathbb{R}^4$ and this induces an action of L^+
 on $\Lambda^k \mathbb{R}^4$ via

$$L \cdot \beta = (\bar{L}^*)^* \beta.$$

We may regard L as a linear mapping from \mathbb{R}^4 to
 and this one obtains an action of L^+ on $\Omega^k \mathbb{R}^4$
 defined by

$$(L \cdot \eta)_{\frac{\partial}{\partial x_i}}(v_1, \dots, v_k) = ((\bar{L}^*)^* \eta)_{\frac{\partial}{\partial x_i}}(v_1, \dots, v_k)$$

$$= \eta_{\frac{\partial}{\partial x_i}}(d\bar{L}^i(v_1), \dots, d\bar{L}^i(v_k))$$

Theorem 1.27 The vacuum Maxwell equations
 $dF = 0$, $dG = 0$, $*F = \mu G$ are invariant
 under this action.

Proof iff $dF = 0$, $dG = 0$, $*F = \mu G$

then

$$d(L \cdot F) = d((\bar{L}^*)^* F) = (\bar{L}^*)^*(dF) = 0$$

$$d(L \cdot G) = d((\bar{L}^*)^* G) = (\bar{L}^*)^*(dG) = 0$$

$$*(L \cdot F) = *((\bar{L}^*)^*(F)) = (\bar{L}^*)^*(*F) = (\bar{L}^*)^*(\mu G) =$$

Theorem 1.28 Let M be a manifold, g a metric on M and μ_g a compatible volume on M . The group of isometries acts on $\Omega^k M$ via

$$\varphi \cdot \eta = (\varphi^{-1})^* \eta$$

for each isometry φ and $\eta \in \Omega^k M$. Then the equations

$$dF = 0, \quad dG = 0, \quad *F = G$$

are invariant under this action.

. Proof $d(\varphi \cdot F) = d(\bar{\varphi}^* F) = \bar{\varphi}^*(dF) = 0$

$$d(\varphi \cdot G) = d(\bar{\varphi}^* G) = \bar{\varphi}^*(dG) = 0$$

$$*(\varphi \cdot F) = *(\bar{\varphi}^* F) = \bar{\varphi}^*(*F) = \bar{\varphi}^* G = \varphi \cdot G.$$

Thus one can find global solutions of Maxwell's equations $dF = 0$ and $\delta F = j$ by finding a solution of the equation

$$\delta(dA) = j$$

and then by setting $F = dA$.

We show how to reformulate this in terms of the Laplace-Beltrami operator Δ defined by

$$\Delta = \delta d + d\delta.$$

Clearly $\Delta A = d(\delta A) + \delta(dA)$ and consequently if $\delta A = 0$ then $\delta(dA) = j$ iff $\Delta A = j$

Theorem 1.2 If A is a solution of the equation $\delta(dA) = j$ then there exist \bar{A} such that $\delta \bar{A} = 0$ and $\Delta \bar{A} = j$. Conversely, if there exists \bar{A} such that $\delta \bar{A} = 0$ and $\Delta \bar{A} = j$ then for $A = \bar{A}$ $\delta(dA) = j$.

Proof Assume A is a solution of $\delta(dA) = j$.

Choose a function f such that $\Delta f = -\delta A$.

Define \bar{A} by $\bar{A} = A + df$. Then

$$\delta \bar{A} = \delta A + \delta(df) = \delta A + df = 0$$

66

and

$$\begin{aligned}\Delta \bar{A} &= (\delta d + d\delta)(\bar{A}) \\ &= \delta(d(\bar{A})) \\ &= \delta(d(A + df)) \\ &= \delta(dA) + \delta(d(df)) \\ &= j.\end{aligned}$$

Conversely, if there exists \bar{A} such that $\delta \bar{A} = 0$ and $\Delta \bar{A} = j$ then $\delta d(\bar{A}) = (\delta d + d\delta)(\bar{A}) = \Delta \bar{A} = j$. \square

Theorem 30 If (U, x) is any chart on Minkowski space then $\delta A = \frac{1}{\sqrt{|g|}} \partial_\mu (A^\mu \sqrt{|g|})$.

Proof Recall that $(*A)_{i_1 i_2 \dots i_m} = \sum_{k_1 k_2 \dots k_m} A^k \sqrt{|g|}$

$$= \sum_{i_1 \dots i_m} (-1)^{j+1} A^j \sqrt{|g|}.$$

Thus $*A = \sum_j (-1)^{j+1} A^j \sqrt{|g|} dx^{i_1} \wedge \dots \wedge dx^{i_m}$

$d(*A) = \sum_j (-1)^{j+1} \partial_k (A^j \sqrt{|g|}) (dx^k \wedge dx^{i_1} \wedge \dots \wedge dx^{i_m})$

$$= \sum_k \partial_k (A^k \sqrt{|g|}) (dx^1 \wedge dx^2 \wedge \dots \wedge dx^n)$$

$$= \frac{1}{\sqrt{|g|}} \partial_\mu (A^\mu \sqrt{|g|}) \nu$$

Where ν is the volume on M . It follows that

$$\delta A = *d(*A) = \frac{1}{\sqrt{|g|}} \partial_\mu (A^\mu \sqrt{|g|}). \quad \square$$

~~Carrollberg~~ $\delta A = \frac{1}{\sqrt{g_1}} \partial_\mu (A^\mu \sqrt{g_1})$

Theorem 31: If x is an inertial chart on Minkowski space and A is a 1-form on M then

$$\Delta A = (\square A)_\nu dx^\nu$$

$$\delta A = \partial_\mu A^\mu$$

Proof: That $\delta A = \partial_\mu A^\mu$ is an immediate consequence of the last theorem ($\sqrt{g_1} = 1$). Now

$$\begin{aligned} (\Delta A)_\nu &= [(\delta d + d \delta) A]_\nu \\ &= (\delta F)_\nu + (d(\delta A))_\nu \\ &= \frac{1}{\sqrt{g_1}} \partial_\mu (\sqrt{g_1} F^{\mu\nu}) g_{\lambda\nu} + d(\delta A)_\nu \end{aligned}$$

From the last theorem, $\delta A = \frac{1}{\sqrt{g_1}} \partial_\mu (A^\mu \sqrt{g_1})$ and

$$d(\delta A) = \partial_\nu \left(\frac{1}{\sqrt{g_1}} \partial_\mu (A^\mu \sqrt{g_1}) \right) dx^\nu$$

Thus

$$d(\delta A)_\nu = \partial_\nu \left(\frac{1}{\sqrt{g_1}} \partial_\mu (A^\mu \sqrt{g_1}) \right)$$

and

$$(\Delta A)_\nu = \frac{1}{\sqrt{g_1}} \partial_\mu (\sqrt{g_1} F^{\mu\nu}) g_{\lambda\nu} + \partial_\nu \left(\frac{1}{\sqrt{g_1}} \partial_\mu (A^\mu \sqrt{g_1}) \right)$$

In inertial coordinates

$$\begin{aligned} (\Delta A)_\nu &= \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) g_{\lambda\nu} + \partial_\nu (\partial_\mu A^\mu) \\ &= \partial_\mu (\partial^\mu A_\nu) - \partial_\mu \partial_\nu A^\mu + \partial_\nu \partial_\mu A^\mu \end{aligned}$$

$$\begin{aligned} &= \partial_\mu (\partial^\mu A_\nu) \\ &= \partial_0^2 A_\nu - \partial_1^2 A_\nu - \partial_2^2 A_\nu - \partial_3^2 A_\nu \\ &= \square(A_\nu) \end{aligned}$$

68