§ XIV.1 THE GAMMA FUNCTION

The Γ - fnct. is the meromorphic extension of the factorial to the entire complex plane. For $Re(z) > 0$

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt$$

Notice $|\Gamma(x+iy)| \leq \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt$, $x > 0$
is absolutely convergent. $\Rightarrow$ $\Gamma(z)$ analytic on right half-plane.

Integrate by parts,

$$\begin{aligned}
\Gamma(z+1) &= \int_0^\infty e^{-t} t^{z} \, dt \\
&= -\int_0^\infty t^{z} \, d(e^{-t}) \\
&= -\cancel{\left[t^{z} e^{-t}\right]\Big|_0^\infty} + z\int_0^\infty e^{-t} t^{z-1} \, dt \\
&= z\,\Gamma(z) \qquad \therefore \boxed{\Gamma(z+1) = z\,\Gamma(z)} \text{ for } re(z) > 0
\end{aligned}$$

Connection with $n!$

$$\Gamma(1) = \int_0^\infty e^{-t} \, dt = -e^{-t}\Big|_0^\infty = -e^{-\infty} + e^{0} = 1.$$

$$\begin{aligned}
\Gamma(2) &= 1 \cdot \Gamma(1) = 1 \\
\Gamma(3) &= 2 \cdot \Gamma(2) = 2 \\
\Gamma(4) &= 3\,\Gamma(3) = 3 \cdot 2
\end{aligned}$$

By induction, suppose $\Gamma(n+1) = n!$ then,

$$\begin{aligned}
\Gamma(n+1+1) &= (n+1)\,\Gamma(n+1) \\
&= (n+1)\,n! \\
&= (n+1)! \qquad \therefore \boxed{\Gamma(n+1) = n! \quad \forall n \in \mathbb{N}}
\end{aligned}$$

Identity principle $\longleftrightarrow$ extends to left-half-plane, how?

Consider, m-fold applications of $\Gamma(z+1) = z\Gamma(z)$,

$$\begin{aligned}\Gamma(z+m) &= \Gamma(z+m-1+1)\\ &= \Gamma(z+m-1)\cdot(z+m-1)\\ &= (z+m-1)\Gamma(z+m-1)\\ &= (z+m-1)(z+m-2)\,\Gamma(z+m-2)\\ &\;\vdots\\ &= (z+m-1)(z+m-2)\cdots(z+1)\,z\,\Gamma(z)\end{aligned}$$

Hence,

$$\Gamma(z) = \frac{\Gamma(z+m)}{(z+m-1)\cdots(z+1)(z)}$$

$\leftarrow$ meromorphic for $Re(z+m) > 0$, $Re(z) > -m$, poles, simple at $0, -1, -2, \ldots, -m+1$.

By identity principle,

Th^m/ The fnct. $\Gamma(z)$ extends to be meromorphic on the entire complex plane where it satisfies $\Gamma(z+1) = z\Gamma(z)$. Its poles are simple poles at $z = 0, -1, -2, \ldots$

<u>Remark</u>: this, upto here, is the easy part. What follows for ∞-products is requiring <u>more</u> mental effort.

INFINITE PRODUCTS & GAMMA FUNCTION

Consider, for $Re(z) > 0$, $n \geq 1$,

$$\Gamma_n(z) = \int_0^n t^{z-1}\left(1 - \frac{t}{n}\right)^n dt \longrightarrow \Gamma(z) \text{ as } n \to \infty$$

Notice $(1 - t/n)^n \leq e^{-t}$ and $(1-t/n)^n \longrightarrow e^{-t}$ for $t \geq 0$

Continuing, let $s = t/n$

$$t = ns \hookrightarrow \left[\begin{array}{ll} t^{z-1} = n^{z-1}s^{z-1} & \begin{pmatrix} t = n \to s = 1 \\ t = 0 \to s = 0 \end{pmatrix} \\ dt = n\,ds & \\ t^{z-1}\left(1 - \frac{t}{n}\right)^n dt = n^{z-1}s^{z-1}(1-s)^n n\,ds & \\ \qquad = n^z s^{z-1}(1-s)^n ds & \end{array}\right]$$

Hence, $\Gamma_n(z) = n^z \int_0^1 s^{z-1}(1-s)^n ds \quad (*)$

$\underline{n=1)}$ $\quad \Gamma_1(z) = \int_0^1 s^{z-1}(1-s)ds = \left(\frac{s^z}{z} - \frac{s^{z+1}}{z+1}\right)\Big|_0^1$

$$\boxed{\Gamma_1(z) = \frac{1}{z} - \frac{1}{z+1} = \frac{1}{z(z+1)}}$$

Next, IBP in $(*)$

$$\Gamma_n(z) = \frac{n^z}{z}\int_0^1 (1-s)^n d(s^z) \quad \leftarrow \text{nice notation !}$$

$$= -\frac{n^z}{z}\int_0^1 s^z d(1-s)^n \quad \leftarrow \text{boundary terms be gone at } s=0 \text{ \& } s=1.$$

$$= \frac{-n^{z+1}}{z}\int_0^1 s^z(1-s)^{n-1}ds \quad \leftarrow d(1-s)^n = n(1-s)^{n-1}(-ds).$$

$$= \left(\frac{n}{n-1}\right)^z \frac{n}{z}\Gamma_{n-1}(z+1)$$

$$\Gamma_n(z) = \frac{n^z n!}{z(z+1)\cdots(z+n-2)}\Gamma_1(z+n-1) = \frac{n^z n!}{z(z+1)\cdots(z+n)}$$

$$\frac{1}{\Gamma_n(z)} = \frac{1}{n^z} z(1+z)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{n}\right)$$

Compare against,

$$G(z) = \prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-z/k}$$

same zeros & multiplicities!

$$\boxed{\text{Th}^{m}/\quad \frac{1}{\Gamma(z)} = z e^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-z/k}}$$

(we established for $Re(z) > 0$, but uniqueness & identity principle extend to $\mathbb{C}$.

Other fun identities

$$-\frac{\Gamma'(z)}{\Gamma(z)} = \frac{1}{z} + \gamma - z\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{z+k}$$

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{k=0}^{\infty}\frac{1}{(z+k)^2}$$