Term-by-Term Calculus for Power Series & Geometric Series Techniques.

The derivative or integral (anti-derivative) of a power series with radius $R > 0$ is a power series with the same radius and coefficients modified in the natural manner:

$$\boxed{\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-x_0)\right) = \sum_{n=1}^{\infty} c_n n (x-x_0)^{n-1}}$$

$$\boxed{\int\left(\sum_{n=0}^{\infty} c_n (x-x_0)\right)dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-x_0)^{n+1}}$$

This theorem allows us to integrate functions by integrating their power series representation. Alternatively, it expands the scope of the geom. series result. We'll see both in the examples given in the next several pages.

$\boxed{E1}$ $\displaystyle\int \frac{dx}{1+x^3} = ?$ Partial fractions is tough here, but,

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty}(-x^3)^n = \sum_{n=0}^{\infty}(-1)^n x^{3n}$$

Thus, for $|x|<1$ ($|-x^3| = |x|^3 < 1$ requires $|x|<1$ aka $x \in (-1,1)$)

$$\int \frac{dx}{1+x^3} = \int\left(\sum_{n=0}^{\infty}(-1)^n x^{3n}\right)dx$$

$$= C + \sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}x^{3n+1}$$

We have, setting aside the "Sigma" notation momentarily,

$$\int \frac{dx}{1+x^3} = \int(1-x^3+x^6-x^9+\cdots)dx = \left(x - \frac{1}{4}x^4 + \frac{1}{7}x^7 + \cdots\right) + C.$$

Power series and Geometric series tricks

$\boxed{E4}$ Identify the function $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \quad -1 \le x \le 1$. Let's use the th$\underline{m}$ to get geom. series,

$$f'(x) = 1 - x^2 + x^4 - \cdots = \frac{a}{1-r} = \frac{1}{1+x^2} \quad \left(\text{where } \begin{array}{l} a=1 \\ r=-x^2 \end{array}\right)$$

Now we can integrate:

$$\int f'(x)\,dx = \int \frac{dx}{1+x^2} \Rightarrow f(x) = c + \tan^{-1}(x)$$

Now $f(0) = 0 \Rightarrow c + \tan^{-1}(0) = 0 \therefore \underline{c = 0}$, hence $\boxed{f(x) = \tan^{-1}(x)}$

> **Remark:** In the last example we were given a series and asked to identify what elementary fnct. it represented, it is more often the case we'll begin with some elementary fnct. and ask for it's power-series expansion. (a fnct that has a power series rep. is called "analytic")

$\boxed{E5}$ Find a power series rep. of $\ln(1+x) = f(x)$. Try diff$\underline{n}$

$$f'(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \quad \left(\begin{array}{l} \text{geom. series} \\ \text{with } a=1 \\ \text{\& } r=-x \end{array}\right)$$

Now integrate term by term,

$$f(x) = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

Find C by eval. f at $x=0$, $f(0) = \ln(1) = \boxed{0 = C}$ thus

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \boxed{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \ln(1+x)}$$

$\boxed{E6}$ Find power series expansion of $f(x) = \frac{1}{(1+x)^2} = \frac{1}{(1+x)^2}$

$$\int \frac{1}{(1+x)^2}\,dx = \frac{-1}{1+x} + C \;,$$

$$\int f(x)\,dx = 2 - \frac{1}{1+x} = 2 - \left(\sum_{n=0}^{\infty} (-x)^n\right) \quad : \text{geom. series} \quad \begin{array}{l} a=1 \\ r=-x \end{array}$$

$$f(x) = \frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\left[2 - \sum_{n=0}^{\infty} (-x)^n\right] = \sum_{n=1}^{\infty} n(-x)^{n-1} = \boxed{1 - 2x + 3x^2 + \cdots = \frac{1}{(1+x)^2}}$$

E7 Power series can also help with some formidable integrals!

$$\int \frac{\tan^{-1}(x)}{x}\,dx = f(x)$$

$$f'(x) = \frac{\tan^{-1}(x)}{x} = \frac{1}{x}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right) = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \dots$$

$$\int f'(x)\,dx = f(x) = C + x - \frac{x^3}{9} + \frac{x^5}{25} - \dots$$

Thus $\boxed{\int \frac{\tan^{-1}(x)}{x}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{(x)^{2n+1}}{(2n+1)^2}}$

<u>Remark:</u> We could approximate definite integrals via their power series expansions,

E8 Find power series expansion of $f(x) = \ln(1+x^2)$. Notice,

$$f'(x) = \left(\frac{1}{1+x^2}\right)2x$$

Now we can see how to apply geometric series result, $a = 2x$, $r = -x^2$

$$f'(x) = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (2x)(-x^2)^n = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}$$

Now recover $f(x)$ by integrating, $f(x) = \int f'(x)\,dx$

$$f(x) = \int \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n 2 \int x^{2n+1}\,dx + C$$

$$= \sum_{n=0}^{\infty} (-1)^n 2\left(\frac{x^{2n+2}}{2n+2}\right) + C \qquad \text{note } f(0) = 0 \Rightarrow \underline{C = 0}$$

Therefore $\boxed{f(x) = \cancel{\frac{2x}{1+x^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2n+2} = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \dots}$

$\ln(1+x^2)$

HOMEWORK : CALCULUS II : §12.9 # 4, 9, 10, 11, 17, 18, 23, 26, 28, 32

(STEWART 6th Ed.)

§12.9 # 4 Find power series expansion for $f(x) = \frac{3}{1-x^4}$

$$f(x) = \frac{3}{1-x^4} = \sum_{n=0}^{\infty} 3(x^4)^n = \boxed{\sum_{n=0}^{\infty} 3x^{4n}}$$

geom. series $a = 3$, $r = x^4$

I.O.C. = (-1,1)

for $x \in (-1,1)$ since $r = x^4$ and $|r| < 1 \Rightarrow |x^4| < 1 \Rightarrow |x| < 1$.

§12.9 # 9 Find power series for $f(x) = \frac{1+x}{1-x}$

$$f(x) = \frac{1+x}{1-x} = \frac{1}{1-x} + \frac{x}{1-x}$$

: use geom. series result with $a = 1$ or $a = x$ and $r = x$.

$$= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x(x)^n$$

$$= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x^{n+1}$$

: let $k = n+1$ thus $n = 0 \Rightarrow k = 1$ and,

$$= 1 + \sum_{n=1}^{\infty} x^n + \sum_{k=1}^{\infty} x^k$$

$$= \boxed{1 + 2\sum_{n=1}^{\infty} x^n}$$

, the I.O.C. = (-1,1) by geometric series since $|r| < 1$ iff $|x| < 1$.

§12.9 #10 Find power series expansion of $f(x) = \frac{x^2}{a^3 - x^3}$

$$f(x) = \frac{x^2}{a^3(1 - x^3/a^3)} = \frac{x^2/a^3}{1 - x^3/a^3}$$

Identify $r = x^3/a^3$ & "a" $= x^2/a^3$ for geom. series,

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{a^3}\left(\frac{x^3}{a^3}\right)^n = \boxed{\sum_{n=0}^{\infty} \frac{x^{3n+2}}{a^{3n+3}}}$$

where $\boxed{\text{I.O.C.} = (-|a|, |a|)}$

since $|r| < 1 \Rightarrow |x^3/a^3| < 1$

$\Rightarrow |x^3| < |a^3|$

$\Rightarrow |x| < |a|$

$\boxed{\S 12.9\#11}$ Use partial fractions to help find power series expansion for $f(x) = \frac{3}{x^2 - x - 2}$

$$f(x) = \frac{3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$\Rightarrow \quad 3 = A(x+1) + B(x-2)$$

$\underline{x=-1}$ $\quad 3 = -3B \quad \therefore \underline{B=-1}$

$\underline{x=2}$ $\quad 3 = 3A \quad \therefore \underline{A=1}$

$|x|<1$ and $\underbrace{\left|\frac{x}{2}\right|<1}_{|x|<2}$

Hence, $f(x) = \frac{-1}{x+1} + \frac{1}{x-2} = \frac{-1}{x+1} - \frac{1}{2(1-x/2)}$

$$\therefore \quad f(x) = -\sum_{n=0}^{\infty}(-x)^n + \sum_{n=0}^{\infty} \frac{-1}{2}\left(\frac{x}{2}\right)^n \leftarrow \text{geom. series result.}$$

$$= \boxed{\sum_{n=0}^{\infty}\left[(-1)^{n+1} - \left(\frac{1}{2}\right)^{n+1}\right]x^n}$$

~~I.O.C. = $\left(\frac{-1}{2}, \frac{1}{2}\right)$~~

I.O.C. = $(-1, 1)$

$\boxed{\S 12.9\#17}$ Find power series expansion for $f(x) = \frac{x^3}{(x-2)^2}$

Notice $g(x) = \frac{1}{(x-2)^2} \longrightarrow \int g(x)dx = \frac{-1}{x-2} \cancel{+C} = \frac{1}{2(1-x/2)} + C$

Hence, $\int g(x)dx = \sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{x}{2}\right)^n \cancel{+C} = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}x^n + C$

$$\therefore \quad g(x) = \frac{d}{dx}\int g(x)dx = \sum_{n=0}^{\infty}\frac{n}{2^{n+1}}x^{n-1}$$

$$\Rightarrow \boxed{f(x) = x^3 g(x) = \sum_{n=0}^{\infty}\frac{n}{2^{n+1}}x^{n+2}}$$

(the R.O.C. is $R=2$)

$\boxed{\S 12.9\#18}$ Find power series for $f(x) = \tan^{-1}(x/3)$ and state the R.O.C. for the series

Notice $f'(x) = \frac{1/3}{1+x^2/9} = \sum_{n=0}^{\infty}\frac{1}{3}\left(-\frac{x^2}{9}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{2n+1}}x^{2n}$ then integrate to get back to $f(x)$,

$$f(x) = \int f'(x)dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^{2n+1}}x^{2n+1} \quad \text{: integrated term by term.}$$

Note $f(0) = \tan^{-1}(0) = 0 = C$ thus,

$$\boxed{\tan^{-1}\left(\frac{x}{3}\right) = \sum \frac{(-1)^n}{2n+1}\left(\frac{1}{3^{2n+1}}\right)x^{2n+1}}$$

Remark: you can substitute $u = \frac{x}{3}$ into $\tan^{-1}(u)$ and get same result.

note the $\boxed{\text{R.O.C.} = 3}$ since $r<1 \Rightarrow \left|\frac{x^2}{3^2}\right| < 1 \rightarrow |x|<3 \rightarrow (-3,3) = \text{I.O.C.}$

$\boxed{\S 12.9 \# 23}$ Calculate a power series sol$^{\underline{n}}$ to $\int \frac{t}{1-t^8}\,dt$, what is the R.O.C.

$$\int\left(\frac{t}{1-t^8}\right)dt = \int\left(\sum_{n=0}^{\infty} t\,(t^8)^n\right)dt$$

: let $a = t$, $r = t^8$

we need $|r| < 1$

hence $|t^8| < 1 \Rightarrow |t| < 1$

thus $\boxed{R.O.C. = 1}$

$$= \int\left(\sum_{n=0}^{\infty} t^{8n+1}\right)dt$$

$$= \boxed{C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}}$$

(integrating or differentiating will not change the R.O.C. I'm always using this fact in the problems of this section.)

$\boxed{\S 12.9 \# 26}$ Calculate $\int \tan^{-1}(x^2)\,dx$ as a power series and find that power series R.O.C.

from $a = 2x$, $r = -x^4$

$\Rightarrow |x| < 1 \therefore \boxed{R.O.C. = 1}$

Following #18, $f(x) = \tan^{-1}(x^2)$

$$f'(x) = \frac{2x}{1+x^4} = \sum_{n=0}^{\infty} 2x(-x^4)^n = \sum_{n=0}^{\infty} 2(-1)^n x^{4n+1}$$

$$f(x) = \int f'(x)\,dx = C + \sum_{n=0}^{\infty} \frac{2(-1)^n}{4n+2} x^{4n+2}$$

Note $f(0) = C = 0 \therefore \tan^{-1}(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{4n+2}$.

Now I can do the integration,

$$\int \tan^{-1}(x^2)\,dx = \int\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{4n+2}\right)dx$$

$$= \boxed{C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(4n+3)} x^{4n+3}}$$

$\boxed{\boxed{\S 12.9 \# 28} \text{ Calculate } \int_0^{0.4} \ln(1+x^4)\,dx \text{ to 6 decimal places}}$

Notice $f(x) = \ln(1+x^4) \Rightarrow \dfrac{df}{dx} = \dfrac{4x^3}{1+x^4} = \sum_{n=0}^{\infty} 4x^3(-x^4)^n = \sum_{n=0}^{\infty} 4(-1)^n x^{4n+3}$

Then $f(x) = \int f'(x)\,dx = c + \sum_{n=0}^{\infty} \dfrac{4(-1)^n}{4n+4} x^{4n+4}$

Observe $f(0) = \ln(1) = 0 = c \quad \therefore f(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n+1} x^{4n+4}$

Thus we can expand the integrand. I'll write the first few terms because I anticipate we'll have an alternating series so I can use the $|S - S_n| \leq b_{n+1}$ error Th^{m},

$$\int_0^{0.4} \ln(1+x^4)\,dx = \int_0^{0.4} \left(x^4 - \tfrac{1}{2}x^8 + \tfrac{1}{3}x^{12} - \tfrac{1}{4}x^{16} + \cdots\right)dx$$

$$= \left[\tfrac{1}{5}x^5 - \tfrac{1}{18}x^9 + \tfrac{1}{39}x^{13} - \tfrac{1}{56}x^{17} + \cdots\right]\Big|_0^{0.4}$$

$$= \underbrace{\tfrac{1}{5}(0.4)^5 - \tfrac{1}{18}(0.4)^9 + \tfrac{1}{39}(0.4)^{13}}_{\text{these certainly give integral to 6 decimals}} - \cdots = \boxed{0.002034\_\_}$$

since $\left|\frac{1}{56}(0.4)^{17}\right| = \underbrace{\frac{4^{17}}{56}}_{3.07\times 10^{-9}}\left(\frac{1}{10}\right)^{17} < 0.0000001$

(keeping 3 terms gets 8 decimals.)

$\boxed{\text{Remark: } b_3 = 1.72\times 10^{-7} \text{ thus } \tfrac{1}{5}(0.4)^5 - \tfrac{1}{18}(0.4)^9 \text{ is enough in fact.}}$

$\boxed{\boxed{\S 12.9\#32} \text{ Show that } f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \text{ is a sol}^{n} \text{ to } y'' + y = 0}$

Notice that

$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2n}{(2n)!} x^{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} x^{2n-1}$$

: note $2n = 0$ when $n = 0$ so we drop $n = 0$.

$$f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!}(2n-1)x^{2n-2}$$

: notice we keep $n = 1$ since $2n - 1 = 2 - 1 = 1 \neq 0$ for the lowest term.

Let's change the index of the $f''(x)$ sum. Also notice $\frac{2n-1}{(2n-1)!} = \frac{1}{(2n-2)!}$

$$f''(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k)!} x^{2k}$$

$\boxed{\begin{array}{l} \text{Let } 2k = 2n-2 \\ n = 1 \Rightarrow 2k = 2-2 = 0 \Rightarrow k = 0 \\ \underline{n = k+1} \end{array}}$

$$= -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$= -f(x) \quad \therefore f''(x) + f(x) = 0 \quad \therefore f(x) \text{ solves } y'' + y = 0$$