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Homework 10, Calculus III

§16.7 #7

$$z = 4 - r^2 = 4 - x^2 - y^2$$

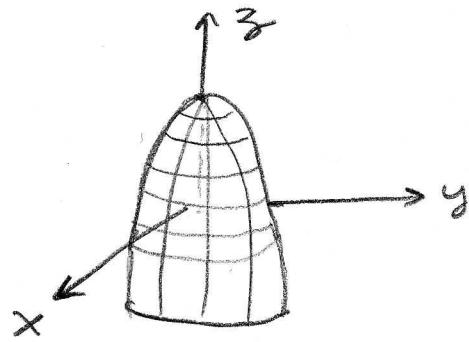
When $x = 0$ we have $z = 4 - y^2$

When $y = 0$ we have $z = 4 - x^2$

This is a paraboloid that opens down
and has top point $(0, 0, 4)$. For
any constant $z = z_0 < 4$ we have

$$4 - z_0 = r^2$$

a circle.

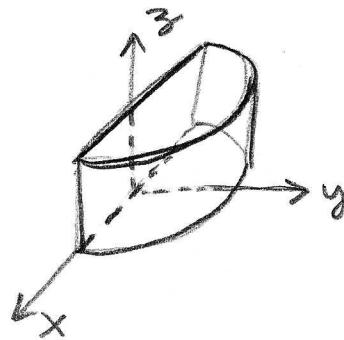


§16.7 #11

$$0 \leq r \leq 2$$

$$-\pi/2 \leq \theta \leq \pi/2$$

$$0 \leq z \leq 1$$



half-cylinder
with height 1
resting on the
xy-plane.

§16.7 #17 E with $x^2 + y^2 \leq 16$ and $-5 \leq z \leq 4$

$$\begin{aligned}
 \iiint_E \sqrt{x^2 + y^2} dV &= \int_0^{2\pi} \int_0^4 \int_{-5}^4 r^2 dr dz d\theta && \leftarrow dV = r dr dz d\theta \\
 &= \int_0^{2\pi} d\theta \int_{-5}^4 dz \int_0^4 r^2 dr \\
 &= (2\pi)(9)(16/3) \\
 &= \boxed{\frac{384\pi}{3}}
 \end{aligned}$$

(since $\left| \frac{\partial(x, y, z)}{\partial(r, z, \theta)} \right| = r$)

$$\S 16.7 \#20 \quad E = \{ (x, y, z) \mid 0 \leq z \leq x+y+5, \quad 4 \leq x^2 + y^2 \leq 9 \} \quad \textcircled{2}$$

We can convert E to cylindrical coordinates via $x = r\cos\theta, y = r\sin\theta$ and of course $z = z$. Observe $x^2 + y^2 = r^2$, E becomes,

$$0 \leq \theta \leq 2\pi$$

$$4 \leq r^2 \leq 9 \Rightarrow 2 \leq r \leq 3$$

$$0 \leq z \leq r\cos\theta + r\sin\theta + 5$$

Thus motivating,

$$\begin{aligned} \iiint_E \mathbf{F} \cdot d\mathbf{V} &= \int_0^{2\pi} \int_2^3 \int_0^{r(\cos\theta + \sin\theta) + 5} (\cos\theta \ r^2) dz \ dr \ d\theta, \quad dV = r \ dz \ dr \ d\theta \\ &= \int_0^{2\pi} \int_2^3 (z|_0^{r(\cos\theta + \sin\theta) + 5}) \cos\theta \ r^2 dr \ d\theta \\ &= \int_0^{2\pi} \int_2^3 [r^3(\cos^2\theta + \sin\theta\cos\theta) + 5r^2\cos\theta] dr \ d\theta \\ &= \int_0^{2\pi} \left[\left(\frac{r^4}{4} \right|_2^3 (\cos^2\theta + \sin\theta\cos\theta) + \left(\frac{5}{3}r^3 \right|_2^3 \cos\theta \right) \right] d\theta \\ &= \int_0^{2\pi} \left[\frac{1}{4}(81 - 16)(\cos^2\theta + \sin\theta\cos\theta) + \frac{5}{3}(27 - 8)\cos\theta \right] d\theta \\ &= \int_0^{2\pi} \underbrace{\left[\frac{65}{4} \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta) + \frac{1}{2}\sin(2\theta) \right) + 25\cos\theta \right]}_{\text{these are periodic functions which are about to be integrated over a whole period.}} d\theta \\ &= \int_0^{2\pi} \frac{65}{8} d\theta \\ &= \frac{65}{8} \theta \Big|_0^{2\pi} \\ &= \boxed{\frac{65\pi}{4}} \end{aligned}$$

By symmetry these integrals are zero.



(this can be a very useful labor saving observation)

§16.7 #26) Find mass m of a ball B which is the set of all points (x, y, z) such that $x^2 + y^2 + z^2 \leq a^2$ where a is some constant and the mass-density (3)

$$\rho = k\sqrt{x^2 + y^2} = kr \quad (\text{proportionality constant})$$

Observe that if $\rho = \frac{dm}{dV}$ then we can add up all the little masses $dm = \rho dV$ by integrating,

$$\begin{aligned} m &= \iiint_B \rho dV \rightarrow \text{notice } x^2 + y^2 + z^2 \leq a^2 \\ &= \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} kr^2 dz dr d\theta \\ &= \int_0^{2\pi} \int_0^a 2kr^2 \sqrt{a^2-r^2} dr d\theta \end{aligned}$$

$$= \int_0^a 4\pi kr^2 \underbrace{\sqrt{a^2-r^2} dr}_{\substack{\text{not an} \\ \text{obvious integral,} \\ \text{prepare for} \\ \text{math battle.}}} \quad : \text{note the integrand is} \\ \text{constant in } \theta \text{ so} \\ \text{we just have the } d\theta \\ \text{integration yield } \Theta \Big|_0^{2\pi} = 2\pi.$$

$$\begin{aligned} \int x^2 \sqrt{a^2 - x^2} dx &= \int (a^2 \sin^2 \theta)(a \cos \theta) a \cos \theta d\theta \quad \left. \begin{array}{l} \text{Let } x = a \sin \theta \\ \text{then } dx = a \cos \theta d\theta \end{array} \right. \\ &= a^4 \int \sin^2 \theta \cos^2 \theta d\theta \quad \left. \begin{array}{l} \sqrt{a^2 - x^2} = \sqrt{a^2 (1 - \sin^2 \theta)} \\ = \sqrt{a^2 \cos^2 \theta} \\ = a \cos \theta. \end{array} \right. \\ &= a^4 \int \frac{-1}{4} (e^{i\theta} - e^{-i\theta})^2 \frac{1}{4} (e^{i\theta} + e^{-i\theta})^2 d\theta \\ &= -\frac{a^4}{16} \int (e^{2i\theta} - 2 + e^{-2i\theta})(e^{2i\theta} + 2 + e^{-2i\theta}) d\theta \\ &= -\frac{a^4}{16} \int (e^{4i\theta} + 2e^{2i\theta} + 1 - 2e^{2i\theta} - 4 - 2e^{-2i\theta} + \\ &\quad \rightarrow + 1 + 2e^{-2i\theta} + e^{-4i\theta}) d\theta \end{aligned}$$

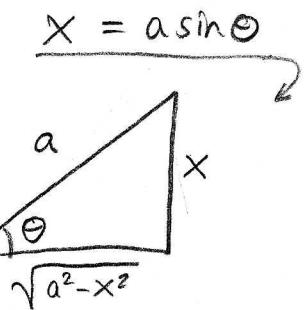
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§ 16.7 #26)

$$\int x^2 \sqrt{a^2 - x^2} dx = \frac{-a^4}{16} \int [2\cos(4\theta) - 2] d\theta$$

$$= \frac{-a^4}{32} \sin(4\theta) + \frac{a^4}{8} \theta$$

$$= \frac{a^4}{32} \sin(4\sin^{-1}\left(\frac{x}{a}\right)) + \frac{a^4}{8} \sin^{-1}\left(\frac{x}{a}\right) +$$



$$\theta = \sin^{-1}\left(\frac{x}{a}\right)$$

Returning to the original integral,

$$\int_0^a 4\pi k r^2 \sqrt{a^2 - r^2} dr = 4\pi k \int_0^{\pi/2} [2\cos(4\theta) - 2] d\theta$$

$0 \leq r \leq a$
 $\Rightarrow r = a\sin\theta$
 bounded by
 $0 \leq \theta \leq \pi/2$

$$= 4\pi k \left(\frac{-a^4}{32} \sin(4\theta) + \frac{a^4}{8} \theta \right) \Big|_0^{\pi/2}$$

$$= 4\pi k \left\{ \frac{-a^4}{32} \sin(2\pi) + \frac{a^4}{8} \frac{\pi}{2} + \frac{a^4}{32} \sin(0) + \frac{a^4}{8}(0) \right\}$$

$$= \frac{\pi^2 k a^4}{4}$$

$$m = \frac{\pi^2 k a^4}{4}$$

Remark: you were given permission to use Mathematica to do this integral.

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§16.7 #26 Use spherical coordinates instead

$$m = \iiint_B \rho dV$$

$$\begin{aligned}
 &= \int_0^{\pi} \int_0^{2\pi} \int_0^a k \sqrt{x^2 + y^2} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
 &= \int_0^{\pi} \int_0^{2\pi} \int_0^a k \sqrt{\rho^2 \cos^2\theta \sin^2\phi + \rho^2 \sin^2\theta \sin^2\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
 &= \int_0^{\pi} \int_0^{2\pi} \int_0^a k \rho^3 \sin^2\phi \, d\rho \, d\theta \, d\phi \\
 &= \int_0^{\pi} 2\pi k \left(\frac{\rho^4}{4} \Big|_0^a \right) \frac{1}{2} (1 - \cos(2\phi)) \, d\phi \\
 &= \frac{\pi a^4 k}{4} \left(\theta - \frac{1}{2} \sin(2\phi) \Big|_0^{\pi} \right) \\
 &= \boxed{\frac{1}{4} k \pi^2 a^4}
 \end{aligned}$$

Coordinate choice matters

This problem has both cylindrical and spherical symmetries associated to it. Often the symmetry of the bounds is the better one to take advantage from. (the integrand suggests cylindricals)