Sol^{n} to hwks from § 8.1 → § 8.4

§ 8.1 #19

$$\lim_{n\to\infty}\left(n^2 e^{-n}\right) = \lim_{n\to\infty}\left(\frac{n^2}{e^n}\right) \underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}} \lim_{n\to\infty}\left(\frac{2n}{e^n}\right) \underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}} \lim_{n\to\infty}\left(\frac{2}{e^n}\right) = \boxed{0}.$$

(extending n to be a continous variable)

§ 8.2 #1 (a.) A sequence is a whole list of numbers. A series (when it exists, aka converges) is just one number.

Remark: sometimes in § 8.5 → 8.9 I'm sloppy and at times say "series" when to be more precise I ought to say "power series". A power series is a function defined pointwise by a series. That means for each x we assign a series $S(x)$.

(b.) Consider the series $S = a_1 + a_2 + a_3 + \cdots \equiv \sum_{n=1}^{\infty} a_n$.
The n^{th} partial sum is S_n,

$$S_n \equiv \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

The series, is the limit of the partial sums,

$$S = \sum_{n=1}^{\infty} a_n \equiv \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty} S_n$$

Then two possibilities occur,

i.) $\lim_{n\to\infty} S_n = S \in \mathbb{R} \iff$ series S convergent.

ii) $\lim_{n\to\infty} S_n$ does not converge. $\iff$ S divergent.

In words, the series converges when the limit of the sequence of partial sums converges. Likewise the series diverges if the limit of the sequence of partial sums diverges.

§ 8.2 #3

$$\sum_{n=1}^{\infty} \frac{12}{(-5)^n} = \frac{-12}{5} + \frac{12}{25} - \frac{12}{125} + \cdots$$

geometric series $a = -12/5$, $r = -1/5$

thus by geom. series result this series converges. Moreover, since it's geometric we know

$$\sum_{n=1}^{\infty} \frac{12}{(-5)^n} = \frac{a}{1-r} = \frac{-12/5}{1+1/5} = \frac{-12}{5}\frac{5}{6} = \boxed{-2}$$

§ 8.2 #5 $\sum_{n=1}^{\infty} \tan(n)$ diverges by n^{th} term test since $\lim_{n\to\infty} \tan(n) \neq 0$.

$\boxed{\begin{array}{l}\S 8.2\\ \#11\end{array}}$ $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1-(-2/3)} = \frac{5}{5/3} = \boxed{3}$

We noticed it is geometric with $a = 5$ and $r = -2/3$.

$\boxed{\begin{array}{l}\S 8.2\\ \#27\end{array}}$ $\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$ (using partial fractions algebra)

$S_n = \left(1 - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \cdots + \left(\cancel{\frac{1}{n-3}} - \cancel{\frac{1}{n-1}}\right) + \left(\cancel{\frac{1}{n-2}} - \frac{1}{n}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right)$

$S_n = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$ ← neat, we've calculated the n^{th} partial sum.

$s = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 + \frac{1}{2} - \underbrace{\frac{1}{n}}_{0} - \underbrace{\frac{1}{n+1}}_{0}\right) = 1 + \frac{1}{2} = \boxed{3/2}$

To summarize, this is a telescoping series that converged to $3/2$.

$\boxed{\begin{array}{l}\S 8.2\\ \#31\end{array}}$ $0.\overline{2} = 0.2222\ldots = \frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \frac{2}{10000} + \cdots = \frac{2/10}{1 - 1/10} = \boxed{\frac{2}{9}}$

Using again the geometric series result with $a = 2/10$, $r = 1/10$.

$\boxed{\begin{array}{l}\S 8.3\\ \#7\end{array}}$ $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a p-series with $p = 4$ $\therefore$ it converges by p-series test.

$\boxed{\begin{array}{l}\S 8.3\\ \#13\end{array}}$ $\sum_{n=1}^{\infty} ne^{-n}$ try integral test. Let $f(x) = xe^{-x}$ then clearly $f(n) = ne^{-n}$. Also $f(x) \geq 0$ for $x \geq 1$

is $f(x)$ decreasing?

$f'(x) = \frac{d}{dx}\left(xe^{-x}\right) = e^{-x} - xe^{-x} = (1-x)e^{-x}$

Thus $f'(x) < 0$ when $x > 1$ hence f is decreasing.

$\int \underbrace{x}_{u}\underbrace{e^{-x}\,dx}_{dv} = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C$ (Used IBP)

$\Rightarrow \int_1^{\infty} xe^{-x}dx = \lim_{t\to\infty}\left(-xe^{-x} - e^{-x}\Big|_1^t\right)$

$= \lim_{t\to\infty}\left(-te^{-t} - \underbrace{e^{-t}}_{0} + e^{-1} + e^{-1}\right)$

$= \lim_{t\to\infty}\left(-t/e^t\right) + 2e^{-1}$ note $\lim_{t\to\infty}\left(\frac{-t}{e^t}\right) \overset{L}{=} \lim_{t\to\infty}\left(\frac{-1}{e^t}\right) = 0.$ $\left(\frac{\infty}{\infty}\right)$

$= 2/e$

Therefore, $\int_1^{\infty} f(x)dx$ converges $\Rightarrow \sum_{n=1}^{\infty} ne^{-n}$ converges.

(using integral test.)

§8.4 #11 $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ is alternating series with $b_n = \frac{1}{n^p}$

Notice that $\frac{1}{n^p} > 0$. But when is it decreasing?

$$\frac{d}{dn}\left(\frac{1}{n^p}\right) = -p\frac{1}{n^{p+1}} < 0 \quad \text{if} \quad p > 0$$

Also if $p > 0$ then $\lim_{n\to\infty}\left(\frac{1}{n^p}\right) = 0$ therefore by the Alternating Series Test the series converges for $p > 0$.

Let $p \leq 0$ then notice,

$$\lim_{n\to\infty}\left(\frac{1}{n^p}\right) \neq 0$$

Therefore by n^{th} term test the series diverges for $p \leq 0$.

§8.4 #13 $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ how many terms needed for $|S - S_n| < 0.00005$.

this is an alternating series with $b_n = 1/n^6$. We use the alternating series estimation theorem,

$$\frac{1}{n^6} = 0.00005 \Rightarrow n = \left(\frac{1}{0.00005}\right)^{1/6} = 5.21$$

Now really I should've looked at $\frac{1}{(n+1)^6}$ but the algebra is easier for n and, now we state,

$$b_6 = \frac{1}{n^6} < 0.00005 \quad \text{for} \quad n \geq 6$$

Thus $|S_5 - S| < b_6 < 0.0005$ using the Alt. Series estimation theorem.

§8.4 #21 A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

For $a_n = (-10)^n/n!$ we find $|a_n| = 10^n/n!$ consider then

$$L = \lim_{n\to\infty}\left|\frac{10^{n+1}}{(n+1)!}\frac{n!}{10^n}\right| = \lim_{n\to\infty}\left(\frac{10}{n+1}\right) = 0 \quad \therefore \sum_{n=1}^{\infty}\frac{10^n}{n!} \text{ converges by ratio test.}$$

$\therefore \sum \frac{(-10)^n}{n!}$ is absolutely convergent.

Remark: absolute convergence is an important concept, but I've not made a point to discuss it in our course. When a series converges but is not absolutely convergent it is said to be conditionally convergent.