

Homework 5, Calculus III

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§15.4 #1 Find tangent plane to $z = 4x^2 - y^2 + 2y$ at $(-1, 2, 4)$.
 Let $z = f(x, y) = 4x^2 - y^2 + 2y$

$$\frac{\partial f}{\partial x} = 8x \quad f_x(-1, 2) = -8$$

$$\frac{\partial f}{\partial y} = -2y + 2 \quad f_y(-1, 2) = -2$$

Thus $\underline{z = f(-1, 2) + f_x(-1, 2)(x+1) + f_y(-1, 2)(y-2)}$

Gives $\boxed{z = 4 - 8(x+1) - 2(y-2)}$

§15.4 #11 Let $f(x, y) = x\sqrt{y}$ is this differentiable at $(1, 4)$?

If so find $L_f(1, 4)$. Observe that

$$\frac{\partial f}{\partial x} = \sqrt{y} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y}}$$

these are both continuous functions near $(1, 4)$ thus we conclude (By Thm (8) on p. 931 which is on 306 of my notes) that f is diff. at $(1, 4)$. This means f has a well-defined tangent plane at $(1, 4)$. That tangent plane is the graph of the linearization $L_f(1, 4)$. As we have discussed linearization at point \leftrightarrow tangent plane at pt.

$$f_x(1, 4) = \sqrt{4} = 2$$

$$f_y(1, 4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

We showed in lecture $L_f(a, b)(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$ thus applying to case $(a, b) = (1, 4)$ yields

$$\boxed{L_f(1, 4)(x, y) = 2 + 2(x-1) + \frac{1}{4}(y-4)}$$

(2)

$$\underline{315.4 \# 39} \quad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

where $R_1 = 25\Omega$, $R_2 = 40\Omega$, $R_3 = 50\Omega$ with errors of 0.5% in each case. Estimate max error in calculated value of R.

Strategy: If $f(x, y, z)$ then $df = f_x dx + f_y dy + f_z dz$ is the total differential of f . This gives us the max error in f to be df based on assumed errors of dx, dy, dz in x, y, z respective. In this problem $x \sim R_1$, $y \sim R_2$, $z \sim R_3$ $f \sim R$.

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

Calculate: for $k = 1, 2, 3$,

$$\begin{aligned} \frac{\partial R}{\partial R_k} &= \frac{-1}{\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^2} \frac{\partial}{\partial R_k} \left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right] \\ &= -R^2 \left[\frac{-1}{R_1^2} \delta_{1k} - \frac{1}{R_2^2} \delta_{2k} - \frac{1}{R_3^2} \delta_{3k} \right] : \frac{\partial R_j}{\partial R_k} = \delta_{jk} \end{aligned}$$

Thus, look at each case and obtain

$$\frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}, \quad \frac{\partial R}{\partial R_2} = \frac{R^2}{R_2^2}, \quad \frac{\partial R}{\partial R_3} = \frac{R^2}{R_3^2}$$

Thus,

$$\begin{aligned} dR &= \frac{\partial R}{\partial R_1} dR_1 + \frac{\partial R}{\partial R_2} dR_2 + \frac{\partial R}{\partial R_3} dR_3 \\ &= R^2 \left[\left(\frac{1}{R_1^2} \right) dR_1 + \left(\frac{1}{R_2^2} \right) dR_2 + \left(\frac{1}{R_3^2} \right) dR_3 \right] \end{aligned}$$

We had R^2 in each $\frac{\partial R}{\partial R_k}$ so I factored it out.

$$\delta_{jk} = \begin{cases} 1 & j=k \\ 0 & j \neq k \end{cases}$$

③

§15.4 #39 (Continued) Notice $\Delta R_1 = 0.005 R_1$, $\Delta R_2 = 0.005 R_2$
 and $\Delta R_3 = 0.005 R_3$ so [using Δ instead of d now
 to emphasize finite increments]

$$\begin{aligned}\Delta R &= R^2 \left(\frac{1}{R_1^2} 0.005 R_1 + \frac{1}{R_2^2} 0.005 R_2 + \frac{1}{R_3^2} 0.005 R_3 \right) \\ &= 0.005 R^2 \left(\underbrace{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}_{1/R} \right) = \underline{0.005 R = \Delta R}\end{aligned}$$

And we know $R_1 = 25\Omega$, $R_2 = 40\Omega$, $R_3 = 50\Omega$ so

$$R = \frac{1}{\frac{1}{25\Omega} + \frac{1}{40\Omega} + \frac{1}{50\Omega}} = 11.76\Omega$$

Thus,

$$\Delta R \approx (0.005)(11.76\Omega) \approx \boxed{0.05882\Omega = \Delta R}$$

Of course we ought to pay significant figures a little respect here so $\boxed{\Delta R = 0.059\Omega}$ which amounts to a max error of $\approx 0.05\%$ in the total resistance.

Remark: So the error of a parallel combination of resistors is the same as the error of the component resistors provided the components have equal uncertainty in their respective value. Also we are speaking of % - error not absolute error.

(4)

§ 15.6 #8 Let $f(x, y) = y^2/x$ and consider $P = (1, 2)$ and the unit vector $\hat{u} = \langle 2/3, \sqrt{5}/3 \rangle$ find

a.) $\nabla f = \langle f_x, f_y \rangle$ thus,

$$\boxed{\nabla f = \langle -y^2/x^2, 2y/x \rangle}$$

b.) $\nabla f(1, 2) = \langle -4/1, 4/1 \rangle = \boxed{\langle -4, 4 \rangle} = \nabla f(1, 2)$

c.) $(D_{\hat{u}} f)(P) = (\nabla f)(P) \cdot \hat{u}$

$$= \langle -4, 4 \rangle \cdot \langle 2/3, \sqrt{5}/3 \rangle$$

$$= -8/3 + 4\sqrt{5}/3$$

$$= \boxed{\frac{1}{3}(4\sqrt{5} - 8)}.$$

§ 15.6 #15 find $(D_{\vec{v}} f)(P)$ at $P = (0, 0, 0)$ in $\vec{v} = \langle 5, 1, -2 \rangle$ direction,

$$f(x, y, z) = xe^y + ye^z + ze^x$$

$$\begin{aligned} \nabla f &= \left\langle \frac{\partial}{\partial x} (xe^y + ye^z + ze^x), \frac{\partial}{\partial y} (xe^y + ye^z + ze^x), \frac{\partial}{\partial z} (xe^y + ye^z + ze^x) \right\rangle \\ &= \langle e^y + ze^x, xe^y + e^z, ye^z + e^x \rangle \end{aligned}$$

Thus we can calculate,

$$(\nabla f)(0, 0, 0) = \langle 1+0, 0+1, 0+1 \rangle = \langle 1, 1, 1 \rangle.$$

Finally, we need to normalize \vec{v} ,

$$\hat{v} = \frac{1}{|\vec{v}|} \vec{v} = \frac{1}{\sqrt{25+1+4}} \langle 5, 1, -2 \rangle = \frac{1}{\sqrt{30}} \langle 5, 1, -2 \rangle.$$

Thus,

$$(D_{\hat{v}} f)(0, 0, 0) = (\nabla f)(0, 0, 0) \cdot \left[\frac{1}{\sqrt{30}} \langle 5, 1, -2 \rangle \right]$$

$$= \frac{1}{\sqrt{30}} \langle 1, 1, 1 \rangle \cdot \langle 5, 1, -2 \rangle = \boxed{\frac{4}{\sqrt{30}}}$$

§ 15.6 #25] Let $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ find the max. rate of change at $(3, 6, -2)$ and the direction in which it occurs. (5)

$$\begin{aligned}\nabla f &= \langle f_x, f_y, f_z \rangle \\ &= \left\langle \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle \\ &= \frac{1}{\sqrt{x^2 + y^2 + z^2}} \langle x, y, z \rangle.\end{aligned}$$

Notice for $(3, 6, -2)$ we have $\sqrt{x^2 + y^2 + z^2} = \sqrt{9 + 36 + 4} = 7$.

$$\nabla f(3, 6, -2) = \frac{1}{7} \langle 3, 6, -2 \rangle.$$

$$|\nabla f(3, 6, -2)| = \frac{1}{7} \sqrt{3^2 + 6^2 + 2^2} = \boxed{1 = |\nabla f(3, 6, -2)|}$$

This occurs in the $\nabla f(3, 6, -2)$ direction which
is the $\frac{1}{7} \langle 3, 6, -2 \rangle$ - direction Max Rate of Change

§ 15.6 #39] Find tangent plane and normal line at $(3, 3, 5)$ for

$$2(x-1)^2 + (y-1)^2 + (z-3)^2 = 10$$

This is level surface of $F(x, y, z) = 2(x-1)^2 + (y-1)^2 + (z-3)^2$
observe that

$$\begin{aligned}\nabla F &= \langle F_x, F_y, F_z \rangle \\ &= \langle 4(x-1), 2(y-1), 2(z-3) \rangle\end{aligned}$$

Thus $\nabla F(3, 3, 5) = \langle 8, 4, 4 \rangle$. As we discussed $\nabla F(P)$ gives us the normal to tangent plane at P for $F = \text{constant}$.

Thus, $8(x-3) + 4(y-3) + 4(z-5) = 0$ tangent plane

Normal line points in direction of normal and goes thru. $(3, 3, 5)$

$$\vec{r}(t) = (3, 3, 5) + t \langle 8, 4, 4 \rangle$$

§15.6 # 53) Are there any points on the hyperboloid $x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$. In other words, does the hyperboloid ever have a tangent plane with normal parallel to $\langle 1, 1, -1 \rangle$?

$$(z = x + y \rightarrow x + y - z = 0, \text{ normal } \langle 1, 1, -1 \rangle)$$

Hyperboloid is level surface $F = 1$ for

$$F(x, y, z) = x^2 - y^2 - z^2 = 1$$

$$\nabla F = \langle 2x, -2y, -2z \rangle$$

We would need some constant k such that

$$\nabla F(x, y, z) = k \langle 1, 1, -1 \rangle \text{ to get that } \nabla F \parallel \langle 1, 1, -1 \rangle$$

$$\langle 2x, -2y, -2z \rangle = \langle k, k, -k \rangle \text{ oops.}$$

$$\begin{array}{l} 2x = k \\ -2y = k \\ -2z = k \end{array} \left. \begin{array}{c} \\ \\ \end{array} \right\} \rightarrow \begin{array}{l} 2x = -2y = -2z \\ -x = -y = -z \end{array}$$

Don't forget the (x, y, z) is on the hyperboloid so

$$x^2 - y^2 - z^2 = 1. \text{ We can solve for } x,$$

$$x^2 - (-x)^2 - (-x)^2 = 1$$

$$-x^2 = 1 \quad \therefore \underline{x = \pm i}$$

No such point exists
on the hyperboloid