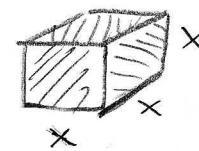


Homework 20 Calculus I

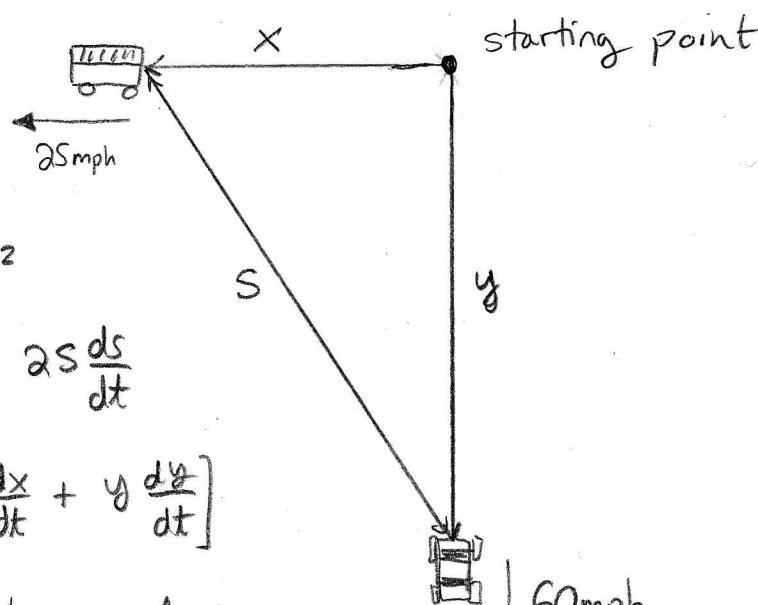
§3.8 #1] How does the rate of change of V and x relate if V is the volume of a cube of side length x ?

$$V = x^3$$

$$\boxed{\frac{dV}{dt} = 3x^2 \frac{dx}{dt}}$$



§3.8 #15] Two cars leave a starting point at the same time. One car goes south at 60 mph while the other travels west at 25 mph. At what rate is the distance between them changing after two hours?



$$x^2 + y^2 = s^2$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2s \frac{ds}{dt}$$

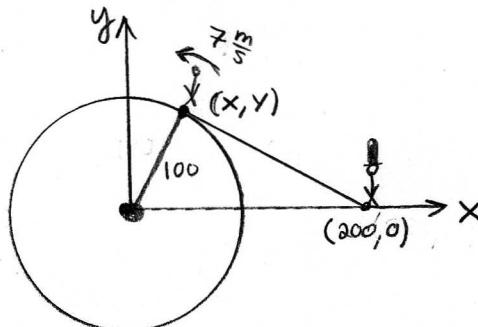
$$\Rightarrow \frac{ds}{dt} = \frac{1}{s} \left[x \frac{dx}{dt} + y \frac{dy}{dt} \right]$$

We assume constant speed so
after two hours $x = (2 \text{ hours}) (25 \frac{\text{miles}}{\text{hour}}) = 50 \text{ miles}$.
and $y = (2 \text{ hours}) (60 \frac{\text{miles}}{\text{hour}}) = 120 \text{ miles}$. Then the
distance between them is $s = \sqrt{50^2 + 120^2} = \sqrt{2500 + 14400} = 130$.

$$\frac{ds}{dt} = \frac{1}{130} [50(25) + 120(60)]$$

$$= \boxed{65 \text{ mph}}$$

§3.8 #43 A runner sprints at 7 m/s around a circular track of radius 100m . Another person looks on from 200m from the center of the track. If the distance between the sprinter and onlooker is 200m then how fast is this distance between them changing? (2)



I choose coordinates at the center of track.

Lets call the distance between X and Q then

$$Q^2 = (200-x)^2 + y^2$$

and we also have that X is on the circle

$$x^2 + y^2 = 100^2$$

Then we can see that x, y, Q are functions of time,

$$2Q \frac{dQ}{dt} = -2(200-x) \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

Our goal is to find dQ/dt . We are given that $Q = 200$ at the time of interest. Lets find x and y for this case,

$$200^2 = (200-x)^2 + y^2$$

$$x^2 + y^2 = 100^2 \rightarrow y^2 = 100^2 - x^2$$

$$\Rightarrow 200^2 = (200-x)^2 + 100^2 - x^2$$

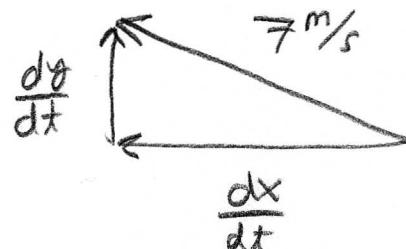
$$\Rightarrow 200^2 = 200^2 - 400x + x^2 + 100^2 - x^2$$

$$\Rightarrow 400x = 100^2$$

$$\Rightarrow \boxed{x = 25}$$

$$\text{Then } y^2 = 100^2 - 25^2 = 9375 \rightarrow \boxed{y = \pm 96.82}$$

We just found $(25, 96.82)$ and $(-25, 96.82)$ are the points on the circle 200 away from the onlooker. We are given the speed is 7 m/s . Notice



$$\vec{v} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 7$$

Now we have $49 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ and

$$0 = x \frac{dx}{dt} + y \frac{dy}{dt} \rightarrow \underline{\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}}$$

Substitute into

$$49 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \frac{x^2}{y^2} \left(\frac{dx}{dt}\right)^2 = \left(1 + \frac{x^2}{y^2}\right) \left(\frac{dx}{dt}\right)^2$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{49}{1 + \frac{x^2}{y^2}} = \frac{49y^2}{y^2 + x^2} = \frac{49y^2}{100^2}$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{49(9375)}{(100)^2}$$

$$\frac{dx}{dt} = \pm \sqrt{\frac{49(9375)}{(100)^2}} = \pm \underline{6.777}$$

$$\frac{dy}{dt} = \pm \frac{25}{96.82} (6.777) = \pm \underline{1.75}$$

The \pm depends on which point we consider and if the runner goes clockwise or counterclockwise. I anticipate there should be two answers one with $dx/dt > 0$ the other with $dx/dt < 0$.

§ 3.8 #43

(4)

$$\begin{aligned}\frac{dQ}{dt} \Big|_{Q=200} &= \frac{1}{Q} \left[-(200-x) \frac{dx}{dt} + y \frac{dy}{dt} \right] \Big|_{Q=200} \\ &= \frac{1}{200} \left[-(200-25)(-6.77) + 96.82(1.75) \right] \\ &= \frac{1}{200} [1184.75 + 169.435] \\ &= \boxed{6.77 \text{ m/s}}\end{aligned}$$

quadrant I
Counterclockwise case.
 $\frac{dx}{dt} < 0$
 $\frac{dy}{dt} > 0$

Or

$$\begin{aligned}\frac{dQ}{dt} \Big|_{Q=200} &= \frac{1}{200} \left[-(200-25)(6.77) + 96.82(-1.75) \right] \quad \text{Quad I. clockwise} \\ &= \frac{1}{200} [-1184.75 - 169.44] \\ &= \boxed{-6.77 \text{ m/s}}\end{aligned}$$

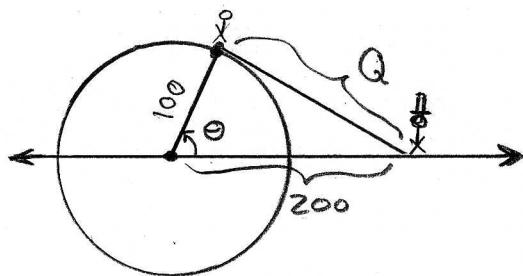
$\frac{dx}{dt} > 0$
 $\frac{dy}{dt} < 0$

My wife Ginny's Sol[®] is much nicer!

See next,

§3.8 #43] Ginny's Sol: Law of Cosines!

(5)



$$Q^2 = 100^2 + 200^2 - 2(100)(200) \cos \theta$$

$$2Q \frac{dQ}{dt} = -4(100)^2 (-\sin \theta \frac{d\theta}{dt})$$

$$\boxed{\frac{dQ}{dt} = \left(\frac{2(100)^2}{Q} \sin \theta\right) \frac{d\theta}{dt}} \quad (*)$$

Now arc length s and radius R and angle θ are related by $s = R\theta$ so $\frac{ds}{dt} = R \frac{d\theta}{dt}$. For our problem $\frac{ds}{dt} = 7 = 100 \frac{d\theta}{dt} \therefore \boxed{\frac{d\theta}{dt} = \frac{7}{100}}$.

When $Q = 200$ what is $\theta = ?$

$$200^2 = 100^2 + 200^2 - 4(100)^2 \cos \theta$$

$$\frac{1}{4} = \cos \theta \rightarrow \sin \theta = \sqrt{1 - \frac{1}{16}} = \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}.$$

Thus we find when $Q = 200$, plugging values into (*),

$$\frac{dQ}{dt} = \left(\frac{2(100)^2}{200} \frac{\sqrt{15}}{4}\right) \frac{7}{100} = \boxed{\frac{7\sqrt{15}}{4} \frac{m}{s}}$$

$$(6.778 \frac{m}{s})$$

(6)

§3.9#1) Find linearization of $f(x) = x^4 + 3x^2$ about $a = -1$.

Notice $f'(x) = 4x^3 + 6x$ thus $f'(-1) = 4(-1)^3 - 6 = -10$.

$$\begin{aligned} L(x) &= f(-1) + f'(-1)(x+1) \\ &= (-1)^4 + 3(-1)^2 - 10(x+1) \\ &= \boxed{4 - 10(x+1) = L(x)} \end{aligned}$$

§3.9#11) Find differential for each function

(a.) $y = x^2 \sin(2x)$

$$\frac{dy}{dx} = 2x \sin(2x) + 2x^2 \cos(2x)$$

$$\therefore \boxed{dy = 2x(\sin(2x) + x\cos(2x))dx}$$

(b.) $y = \sqrt{1+t^2}$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{1+t^2}} \frac{d}{dt}(1+t^2) = \frac{t}{\sqrt{1+t^2}}$$

$$\therefore \boxed{dy = \frac{t}{\sqrt{1+t^2}} dt}$$

Remark: You can think of these as a short-hand for the derivative.

§3.9#19) Let $x = 2$, $\Delta x = -0.4$ compute Δy and "dy" for the given values of x and $\Delta x = dx$ for $y = \underbrace{2x - x^2}_{f(x)}$. Notice $x_1 = 2$, $x_2 = 1.6$ thus

$$y_1 = 2(2) - 2^2 = 0 \quad \& \quad y_2 = 2(1.6) - (1.6)^2 = 0.64.$$

Consequently,

$$\Delta y = y_2 - y_1 = 0.64 - 0 = \boxed{0.64 = \Delta y}$$

Whereas "dy" aka $L(x_2) - L(x_1) = f'(x_1)\Delta x$

Notice $f'(x) = 2 - 2x$ thus $f'(x_1) = 2 - 4 = -2$.

$$\text{"dy"} = -2(-0.4) = \boxed{0.8 = \text{"dy"}}$$

Remark: I object. dy is infinitesimal. This is bad notation.