

Homework 8, Calculus III

(1)

§16.9 #1 Find Jacobian of the following transformation

$$x = 5u - v \Rightarrow x_u = 5, x_v = -1$$

$$y = u + 3v \Rightarrow y_u = 1, y_v = 3.$$

$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} &= \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \quad \text{→ "Jacobian" is determinant of Jacobian matrix.} \\ &= \det \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix} \\ &= |5+1| \\ &= \boxed{16}. \end{aligned}$$

Remark: $F(u, v) = (x(u, v), y(u, v))$
 then $DF = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$
 so $\frac{\partial(x, y)}{\partial(u, v)} = \det(DF)$

§16.9 #5 Let $x = u/v, y = v/w, z = w/u$ then

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(u, v, w)} &= \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} \\ &= \det \begin{bmatrix} 1/v & -u/v^2 & 0 \\ 0 & 1/w & -v/w^2 \\ -w/u^2 & 0 & 1/u \end{bmatrix} \\ &= \frac{1}{v} \left(\frac{1}{w} \frac{1}{u} + \frac{v}{w^2}(0) \right) + \frac{u}{v^2} \left(0 \left(\frac{1}{u} \right) - \frac{v}{w^2} \left(\frac{w}{u^2} \right) \right) \\ &= \frac{1}{uvw} - \frac{uvw}{u^2v^2w^2} \\ &= \boxed{0}. \end{aligned}$$

Remark: This transformation would not be allowed if we wish to use the change of variables integration Th^{*}. That Th^{*} insists that Jacobian $\neq 0$.

§ 16.9 #7 (You are not responsible for the proof/Remark in Calc III). ②

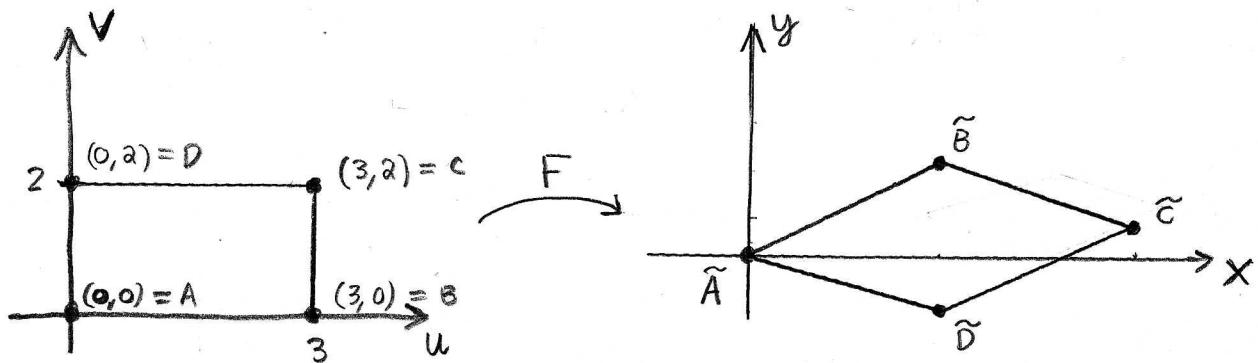
Let $S = \{(u, v) \mid 0 \leq u \leq 3, 0 \leq v \leq 2\}$

Suppose $F(u, v) = (x(u, v), y(u, v))$ where

$$x(u, v) = 2u + 3v$$

$$y(u, v) = u - v$$

Find the image of S under the transformation F .



We can prove that a nonzero linear transformation will map lines to lines thus if we figure out where the corners map to then we can just connect the dots.

$$F(0, 0) = (0, 0) = \tilde{A}$$

$$F(3, 0) = (6, 3) = \tilde{B}$$

$$F(3, 2) = (6+6, 3-2) = (12, 1) = \tilde{C}$$

$$F(0, 2) = (6, -2) = \tilde{D}$$

Remark: to make transformation maintain # of distinct corners we need $\det(A) \neq 0$

Proof: Suppose $F(u, v) = (au + bv, cu + dv)$ this is a linear transformation of $\mathbb{R}^2 \rightarrow \mathbb{R}^2$. Using matrix notation,

$$F(u, v) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Or we may simply write $F(\vec{u}) = A\vec{u}$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} u \\ v \end{bmatrix}$. Take a parametrically described line in uv -space it will be $\vec{u}(t) = \vec{u}_0 + t\vec{v}_0$ generically. Observe

$$F(\vec{u}(t)) = A(\vec{u}_0 + t\vec{v}_0) = (A\vec{u}_0) + t(A\vec{v}_0)$$

This is a line $\vec{x}(t) = \vec{x}_0 + t\vec{y}_0$ where $\vec{x}_0 = A\vec{u}_0$ and $\vec{y}_0 = A\vec{v}_0$. We need to assume $\det(A) \neq 0$ keep F one-one mapping.

(3)

$$\S 16.9 \#13 \quad \text{Let } R = \{(x, y) \mid 9x^2 + 4y^2 \leq 36\}$$

Let $x = 2u$ and $y = 3v$ thus we'll change the ellipse to

$$36 = 9x^2 + 4y^2 = 9(2u)^2 + 4(3v)^2 \\ = 36u^2 + 36v^2 \rightarrow \boxed{u^2 + v^2 = 1}$$

Go from ellipse in xy -plane to unit circle in uv -space

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6.$$

Thus calculate, let $S = \{(u, v) \mid u^2 + v^2 \leq 1\}$

$$\iint_R x^2 dA = \iint_S (2u)^2 \frac{\partial(x, y)}{\partial(u, v)} du dv$$

$$= \iint_S 24u^2 du dv$$

$$= \int_0^1 \int_0^{2\pi} 24r^2 \cos^2 \theta r dr d\theta$$

$$= \int_0^1 24r^3 dr \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta) d\theta$$

$$= (6r^4 \Big|_0^1) \left(\frac{1}{2}(\theta + \frac{1}{2}\sin(2\theta)) \Big|_0^{2\pi} \right)$$

$$= (6 - 0) \left[\frac{1}{2}(2\pi + \cancel{\frac{\sin(4\pi)}{2}}) - \frac{1}{2}(0 + \cancel{\frac{\sin(0)}{2}}) \right]$$

$$= \boxed{6\pi}$$

let $u = r\cos\theta$
 $v = r\sin\theta$
then S becomes
 $0 \leq \theta \leq 2\pi$ and
 $0 \leq r \leq 1$ in
 rv -space.

§16.9 #17a Let $E = \{ (x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \}$ calculate the volume of E by making the change of coordinates

$$x = au$$

$$y = bv$$

$$z = cw$$

under this change of coordinates E morphs from an ellipsoid to a ball B ;

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = (au)^2/a^2 + (bv)^2/b^2 + (cw)^2/c^2 \\ = u^2 + v^2 + w^2$$

That is $B = \{ (u, v, w) \mid u^2 + v^2 + w^2 \leq 1 \}$

$$\iiint_E dV = \iiint_E dx dy dz$$

$$= \iiint_B \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw \quad : \quad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

$$= \iiint_B abc du dv dw$$

$$= abc \iiint_B du dv dw$$

$$\text{Volume of unit sphere} = \frac{4}{3}\pi.$$

$$= \boxed{\frac{4\pi abc}{3}}$$

Remark: the volume of the unit sphere is calculated as follows,

$$\iiint_B du dv dw = \int_0^1 \int_0^\pi \int_0^{2\pi} \left| \frac{\partial(u, v, w)}{\partial(\rho, \theta, \phi)} \right| d\theta d\phi d\rho \quad : \quad u = \rho \cos\theta \sin\phi \\ v = \rho \sin\theta \sin\phi \\ w = \rho \cos\phi$$

Calculate the Jacobian:

$$\frac{\partial(u, v, w)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} u_\rho & u_\theta & u_\phi \\ v_\rho & v_\theta & v_\phi \\ w_\rho & w_\theta & w_\phi \end{vmatrix} = \begin{vmatrix} \cos\theta \sin\phi & -\rho \sin\theta \sin\phi & \rho \cos\theta \cos\phi \\ \sin\theta \sin\phi & \rho \cos\theta \sin\phi & \rho \sin\theta \cos\phi \\ \cos\phi & 0 & -\rho \sin\phi \end{vmatrix} =$$

$$= -\rho^2 \cos^2 \theta \sin^3 \phi + \rho^2 \sin \theta \sin \phi [-\sin^2 \theta \sin \phi - \cos^2 \theta \sin \phi] - \rho^2 \cos^2 \theta \cos^2 \phi \sin \phi \\ = -\rho^2 \cos^2 \theta \sin \phi - \rho^2 \sin^2 \theta \sin \phi \\ = -\rho^2 \sin \phi$$

$$\text{Thus, } \iiint_B du dv dw = \int_0^1 \int_0^\pi \int_0^{2\pi} \rho^2 \sin \phi d\theta d\phi d\rho = 2\pi \left(\frac{\rho^3}{3} \Big|_0^1 \right) (-\cos \phi \Big|_0^\pi) = \frac{4\pi}{3}.$$