

§14.2 #5] Sketch curve, find  $\vec{r}'(t)$ , plot  $\vec{r}(\pi/4)$  and  $\vec{r}'(\pi/4)$  for  $\vec{r}(t) = \langle \sin t, 2\cos t \rangle$ .  
 The sketch is easiest if we notice  $x = \sin t$   $y = 2\cos t$  parametrize an ellipse since  $\sin^2 t + \cos^2 t = x^2 + (\frac{y}{2})^2 = 1$ .

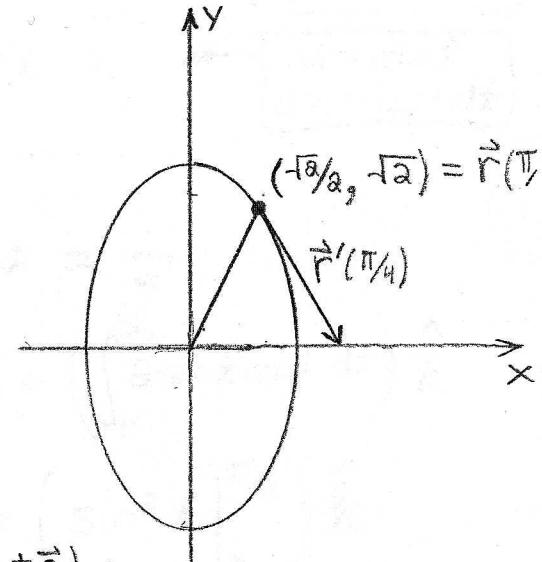
$$\vec{r}(t) = \langle \sin t, 2\cos t \rangle$$

$$\vec{r}'(t) = \langle \cos t, -2\sin t \rangle$$

Evaluate at  $t = \pi/4$  to find,

$$\vec{r}(\pi/4) = \left\langle \frac{\sqrt{2}}{2}, 2\frac{\sqrt{2}}{2} \right\rangle = \left\langle \frac{\sqrt{2}}{2}, \sqrt{2} \right\rangle$$

$$\vec{r}'(\pi/4) = \left\langle \frac{\sqrt{2}}{2}, -2\frac{\sqrt{2}}{2} \right\rangle = \left\langle \frac{\sqrt{2}}{2}, -\sqrt{2} \right\rangle$$



§14.2 #16] Calculate the derivative of  $\vec{r}(t) = t\vec{a} \times (\vec{b} + t\vec{c})$  where  $\vec{a}, \vec{b}, \vec{c}$  are constant vectors.

$$\begin{aligned} \frac{d}{dt}(\vec{r}(t)) &= \frac{d}{dt} [t\vec{a} \times (\vec{b} + t\vec{c})] \\ &= \frac{d}{dt}[t\vec{a}] \times [\vec{b} + t\vec{c}] + t\vec{a} \times \left[ \frac{d}{dt}[\vec{b} + t\vec{c}] \right] \\ &= \left( \frac{dt}{dt}\vec{a} + t \cancel{\frac{d}{dt}(\vec{a})} \right) \times [\vec{b} + t\vec{c}] + t\vec{a} \times \left[ \cancel{\frac{d\vec{b}}{dt}} + \frac{dt}{dt}\vec{c} + t \cancel{\frac{d\vec{c}}{dt}} \right] \\ &= \vec{a} \times [\vec{b} + t\vec{c}] + t\vec{a} \times \vec{c} \\ &= \boxed{\vec{a} \times [\vec{b} + 2t\vec{c}]} \end{aligned}$$

Alternatively could observe  $\vec{r}(t) = \vec{a} \times (t\vec{b} + t^2\vec{c})$  thus

$$\begin{aligned} \frac{d}{dt}[\vec{r}(t)] &= \frac{d}{dt}[\vec{a} \times (t\vec{b} + t^2\vec{c})] \\ &= \vec{a} \times \left[ \frac{d}{dt}(t\vec{b} + t^2\vec{c}) \right] \\ &= \vec{a} \times \left[ \frac{d}{dt}(t)\vec{b} + \frac{d}{dt}(t^2)\vec{c} \right] \\ &= \boxed{\vec{a} \times (\vec{b} + 2t\vec{c})} \end{aligned}$$

- we can pull constant vectors out of derivatives
- I showed why in the first calculation

(2)

§14.2 #35 To begin let me remind you that,

$$\int 3 \sin^2 t \cos t dt = \int 3u^2 du = u^3 + C = \sin^3 t + C \quad \begin{array}{|l} U = \sin t \\ du = \cos t dt \end{array}$$

$$\int 3 \sin t \cos^2 t dt = - \int 3u^2 du = -u^3 + C = -\cos^3 t + C \quad \begin{array}{|l} U = \cos t \\ du = -\sin t dt \end{array}$$

$$\int 2 \sin t \cos t dt = \int 2u du = u^2 + C = \sin^2 t + C \quad \begin{array}{|l} U = \sin t \\ du = \cos t dt \end{array}$$

With these in mind, integrate

$$\int_0^{\pi/2} [3 \sin^2 t \cos t \hat{i} + 3 \sin t \cos^2 t \hat{j} + 2 \sin t \cos t \hat{k}] dt = ?$$

$$? = \left( \int_0^{\pi/2} 3 \sin^2 t \cos t dt \right) \hat{i} + \left( \int_0^{\pi/2} 3 \sin t \cos^2 t dt \right) \hat{j} + \left( \int_0^{\pi/2} 2 \sin t \cos t dt \right) \hat{k}$$

$$= (\sin^3 t \Big|_0^{\pi/2}) \hat{i} + (-\cos^3 t \Big|_0^{\pi/2}) \hat{j} + (\sin^2 t \Big|_0^{\pi/2}) \hat{k}$$

$$= (\sin^3(\pi/2) - \sin^3(0)) \hat{i} + (-\cos^3(\pi/2) + \cos^3(0)) \hat{j} + (\sin^2(\pi/2) - \sin^2(0)) \hat{k}$$

$$= \hat{i} + \hat{j} + \hat{k}$$

$$= \boxed{\langle 1, 1, 1 \rangle}$$

§14.4 #9 Let  $\vec{r}(t) = \langle t^2+1, t^3, t^2-1 \rangle$  represent the position of something

Velocity :  $\vec{v}(t) = \vec{r}'(t) = \langle 2t, 3t^2, 2t \rangle$ .

acceleration :  $\vec{a}(t) = \vec{v}'(t) = \langle 2, 6t, 2 \rangle$ .

Speed :  $\frac{ds}{dt} = |\vec{v}(t)| = \sqrt{4t^2 + 9t^4 + 4t^2}$

(3)

§14.4 #15 Suppose  $\vec{a} = \langle 1, 1, 0 \rangle$  and  $\vec{v}(0) = \langle 0, 0, 1 \rangle$ ,  $\vec{r}(0) = \langle 1, 0, 0 \rangle$ . Find the velocity and position vectors of the particle.

$$\vec{a} = \frac{d\vec{v}}{dt} \rightarrow \int \vec{a} dt = \int \frac{d\vec{v}}{dt} dt = \vec{v}(t).$$

$$\vec{v}(t) = \int \langle 1, 1, 0 \rangle dt = \langle t + c_1, t + c_2, c_3 \rangle$$

$$\vec{v}(0) = \langle 0, 0, 1 \rangle = \langle c_1, c_2, c_3 \rangle$$

$$\Rightarrow c_1 = 0, c_2 = 0, c_3 = 1$$

$$\boxed{\vec{v}(t) = \langle t, t, 1 \rangle}$$

$$\vec{v} = \frac{d\vec{r}}{dt} \rightarrow \int \vec{v} dt = \int \frac{d\vec{r}}{dt} dt = \vec{r}(t)$$

$$\vec{r}(t) = \int \langle t, t, 1 \rangle dt = \left\langle \frac{t^2}{2} + c_1, \frac{t^2}{2} + c_2, t + c_3 \right\rangle$$

$$\vec{r}(0) = \langle 1, 0, 0 \rangle = \langle c_1, c_2, c_3 \rangle$$

$$\Rightarrow c_1 = 1, c_2 = 0, c_3 = 0$$

$$\boxed{\vec{r}(t) = \langle 1 + t^2/2, t^2/2, t \rangle}$$

There is another method to integrate, I actually prefer the arguments that follow,

$$\int_0^t \vec{a}(\tau) d\tau = \int_0^t \frac{d}{d\tau} (\vec{v}(\tau)) d\tau = \int_0^t \langle 1, 1, 0 \rangle d\tau$$

$$\vec{v}(t) - \vec{v}(0) = \langle t, t, 0 \rangle$$

$$\therefore \vec{v}(t) = \vec{v}(0) + \langle t, t, 0 \rangle$$

$$\boxed{\vec{v}(t) = \langle t, t, 1 \rangle}$$

$$\int_0^t \vec{v}(\tau) d\tau = \underbrace{\int_0^t \frac{d}{d\tau} (\vec{r}(\tau)) d\tau}_{\vec{r}(t) - \vec{r}(0)} = \int_0^t \langle \tau, \tau, 1 \rangle d\tau$$

$$\vec{r}(t) - \vec{r}(0) = \langle t^2/2, t^2/2, t \rangle$$

$$\boxed{\vec{r}(t) = \langle t^2/2 + 1, t^2/2, t \rangle}$$

§ 14.4 #35

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

$$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\vec{r}''(t) = \langle -\cos t, -\sin t, 0 \rangle = \vec{a}(t)$$

The unit tangent vector,

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{1}{\sqrt{\sin^2 t + \cos^2 t + 1}} \langle -\sin t, \cos t, 1 \rangle = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$$

$$\vec{T}'(t) = \frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle$$

$$\|\vec{T}'(t)\| = \frac{1}{\sqrt{2}} \sqrt{\cos^2 t + \sin^2 t} = \frac{1}{\sqrt{2}}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \langle -\cos t, -\sin t, 0 \rangle.$$

Now it's a simple matter to calculate

$$\begin{aligned} a_T &= \vec{a} \cdot \vec{T} = \langle -\cos t, -\sin t, 0 \rangle \cdot \left( \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle \right) \\ &= \frac{1}{\sqrt{2}} (\cos t \sin t - \sin t \cos t) \end{aligned}$$

$$a_T = 0$$

$$\begin{aligned} a_N &= \vec{a} \cdot \vec{N} = \langle -\cos t, -\sin t, 0 \rangle \cdot \langle -\cos t, -\sin t, 0 \rangle \\ &= \cos^2 t + \sin^2 t \end{aligned}$$

$$a_N = 1$$

These accelerations describe a motion in a circle which moves with constant velocity. The acceleration describes a change in direction not magnitude for the velocity. Notice  $\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$  would have the same  $a_T$  and  $a_N$ . The  $z'=t$  just describes a constant velocity motion along  $z$ .