

$$\boxed{\text{§15.3\#22}} \quad \text{Let } f(x, y) = x^y$$

$$f_x = \frac{\partial}{\partial x} (x^y) = \boxed{y x^{y-1} = f_x}$$

$$f_y = \frac{\partial}{\partial y} (x^y) = \boxed{\ln(x) x^y = f_y} \quad (\text{for } x > 0)$$

When  $x < 0$   $x^y$  is not continuous, much less differentiable.  
Try to graph  $y = (-2)^x$ . see what happens.

$$\boxed{\text{§15.3\#39}} \quad \text{Let } f(x, y) = \ln(x + \sqrt{x^2 + y^2}) \quad \text{find } f_x(3, 4)$$

$$\begin{aligned} f_x &= \frac{\partial}{\partial x} \left[ \ln(x + \sqrt{x^2 + y^2}) \right] \\ &= \frac{1}{x + \sqrt{x^2 + y^2}} \frac{\partial}{\partial x} \left[ x + \sqrt{x^2 + y^2} \right] \\ &= \frac{1}{x + \sqrt{x^2 + y^2}} \left( 1 + \frac{1}{2\sqrt{x^2 + y^2}} \frac{\partial}{\partial x} (x^2 + y^2) \right) \\ &= \frac{1}{x + \sqrt{x^2 + y^2}} \left( 1 + \frac{x}{\sqrt{x^2 + y^2}} \right) \end{aligned}$$

$$\text{Thus, } f_x(3, 4) = \frac{1}{3 + \sqrt{9 + 16}} \left( 1 + \frac{3}{\sqrt{9 + 16}} \right) = \frac{1}{8} \left[ \frac{5}{5} + \frac{3}{5} \right]$$

$$\therefore \boxed{f_x(3, 4) = \frac{1}{5}}$$

§ 15.3 # 47

find  $\partial z / \partial x$  and  $\partial z / \partial y$ ,

$$x - z = \tan^{-1}(yz)$$

Differentiate with respect to  $x$  (hold  $y$  fixed),

$$\frac{\partial}{\partial x} (x - z) = \frac{\partial}{\partial x} [\tan^{-1}(yz)]$$

$$1 - \frac{\partial z}{\partial x} = \frac{1}{1+(yz)^2} \frac{\partial}{\partial x} [yz] = \left[ \frac{y}{1+y^2z^2} \right] \frac{\partial z}{\partial x}$$

$$1 = \left[ 1 + \frac{y}{1+y^2z^2} \right] \frac{\partial z}{\partial x} \quad \therefore \frac{\partial z}{\partial x} = \frac{1}{1 + \frac{y}{1+y^2z^2}}$$

Likewise,

$$\frac{\partial}{\partial y} (x - z) = \frac{\partial}{\partial y} [\tan^{-1}(yz)]$$

$$-\frac{\partial z}{\partial y} = \frac{1}{1+y^2z^2} \frac{\partial}{\partial y} [yz]$$

$$-\frac{\partial z}{\partial y} = \frac{1}{1+y^2z^2} \left[ z + y \frac{\partial z}{\partial y} \right]$$

$$-\frac{z}{1+y^2z^2} = \frac{\partial z}{\partial y} + \frac{y}{1+y^2z^2} \frac{\partial z}{\partial y}$$

$$\therefore \frac{\partial z}{\partial y} = \left( \frac{1}{1 + \frac{y}{1+y^2z^2}} \right) \left( \frac{-z}{1+y^2z^2} \right)$$

$$\frac{\partial z}{\partial y} = \frac{-z}{1+y^2z^2+y}$$

could simplify a bit more,

$$z_x = \frac{1+y^2z^2}{1+y^2z^2+y}$$

§ 15.3 # 50

Find  $z_x, z_y$  for

a.)  $z = f(x)g(y)$

$$\frac{\partial}{\partial x} (f(x)g(y)) = \frac{\partial}{\partial x} (f(x))g(y) = \frac{df}{dx}g = z_x$$

$$\frac{\partial}{\partial y} (f(x)g(y)) = f(x) \frac{\partial}{\partial y} (g(y)) = f \frac{dg}{dy} = z_y$$

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$$\left( \frac{\partial f}{\partial x} = f'(xy) \right)$$

b.)  $z = f(xy)$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} \frac{\partial}{\partial x}(xy) = \boxed{y \frac{\partial f}{\partial x} = z_x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} \frac{\partial}{\partial y}(xy) = \boxed{x \frac{\partial f}{\partial y} = z_y}$$

c.)  $z = f(x/y)$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} \frac{\partial}{\partial x} \left( \frac{x}{y} \right) = \boxed{\frac{1}{y} \frac{\partial f}{\partial x} = z_x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} \frac{\partial}{\partial y} \left( \frac{x}{y} \right) = \boxed{\frac{-x}{y^2} \frac{\partial f}{\partial y} = z_y}$$

§15.3#51/  $f(x,y) = x^3y^5 + 2x^4y$  find  $f_{x_i x_j}$   $i,j = x,y$

$$f_x = 3x^2y^5 + 8x^3y \quad f_y = 5x^3y^4 + 2x^4$$

$$\begin{aligned} f_{xx} &= 6xy^5 + 24x^2y & f_{yy} &= 20x^3y^3 \\ f_{xy} &= 15x^2y^4 + 8x^3 & f_{yx} &= 15x^2y^4 + 8x^3 \end{aligned}$$

Of course by Clairaut's Th<sup>m</sup> could have anticipated that  $f_{xy} = f_{yx}$ .

§15.5#1/  $x = \sin t$ ,  $y = e^t$ ,  $z = x^2 + y^2 + xy$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$= (2x + y) \cos t + (2y + x) e^t$$

$$= (2 \sin t + e^t) \cos t + (2e^t + \sin t) e^t$$

$$= \boxed{2 \sin t \cos t + e^t (\cos t + \sin t + 2e^t)} = \frac{dz}{dt}$$

is this better than the line above? Its debatable.

§ 15.5 # 7

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$$\begin{aligned} z &= x^2 y^3 & z_x &= 2xy^3 & z_y &= 3x^2 y^2 \\ x &= s \cos t & x_s &= \cos t & x_t &= -s \sin t \\ y &= s \sin t & y_s &= \sin t & y_t &= s \cos t \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \\ &= (2xy^3)(-s \sin t) + (3x^2 y^2)(s \cos t) \\ &= -2(s \cos t)(s \sin t)^4 + 3(s \sin t)^2 (s \cos t)^3 \\ &= \boxed{s^5 [-2 \cos t \sin^4 t + 3 \sin^2 t \cos^3 t]} = \frac{\partial z}{\partial t} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\ &= (2xy^3) \cos t + (3x^2 y^2) \sin t \\ &= 2(s \cos t)(s \sin t)^3 \cos t + 3(s \cos t)^2 (s \sin t)^2 \sin t \\ &= \boxed{s^2 [2 \cos^2 t \sin^3 t + 3 \cos^2 t \sin^3 t]} = \frac{\partial z}{\partial s} \end{aligned}$$

§ 15.5 # 11  $z = e^r \cos \theta$  with  $r = st$ ,  $\theta = \sqrt{s^2 + t^2}$

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial t} \\ &= (e^r \cos \theta) s - e^r \sin \theta \left( \frac{t}{\sqrt{s^2 + t^2}} \right) \\ &= \boxed{e^{st} \left[ s \cos \sqrt{s^2 + t^2} - \frac{t}{\sqrt{s^2 + t^2}} \sin \sqrt{s^2 + t^2} \right]} = \frac{\partial z}{\partial t} \end{aligned}$$

$$\frac{\partial z}{\partial s} = (e^r \cos \theta) t - (e^r \sin \theta) \frac{s}{\sqrt{s^2 + t^2}} \quad \leftarrow \text{dangerous}$$

$$\boxed{\frac{\partial z}{\partial s} = e^{st} \left( t \cos \sqrt{s^2 + t^2} - \frac{s}{\sqrt{s^2 + t^2}} \sin \sqrt{s^2 + t^2} \right)}$$

not partial credit friendly work.

$$z = x^2 + xy^3$$

$$x = uv^2 + w^3$$

$$y = u + ve^w$$

Find  $z_u, z_v, z_w$  when  $u=2, v=1, w=0$   
 Let's see we'll need to know

$$\frac{\partial x}{\partial u} = v^2 + w^3 \quad x_u(2,1,0) = 1$$

$$\frac{\partial x}{\partial v} = 2uv \quad x_v(2,1,0) = 4$$

$$\frac{\partial x}{\partial w} = 3w^2 \quad x_w(2,1,0) = 0$$

$$\frac{\partial y}{\partial u} = 1 \quad y_u(2,1,0) = 1$$

$$\frac{\partial y}{\partial v} = e^w \quad y_v(2,1,0) = e^0 = 1$$

$$\frac{\partial y}{\partial w} = ve^w \quad y_w(2,1,0) = 1e^0 = 1.$$

Also notice  $x(2,1,0) = 2$  and  $y(2,1,0) = 3$ .

$$\text{and } z_x(2,1,0) = 2x + y^3 = 4 + 27 = 31.$$

$$z_y(2,1,0) = 3xy^2 = 3(2)(9) = 54.$$

Finally at the point  $(2,1,0)$  in  $uvw$ -space,

$$\frac{\partial z}{\partial u} = z_x \frac{\partial x}{\partial u} + z_y \frac{\partial y}{\partial u} = (31) + (54) = 85 = \frac{\partial z}{\partial u} \Big|_{(2,1,0)}$$

$$\frac{\partial z}{\partial v} = z_x \frac{\partial x}{\partial v} + z_y \frac{\partial y}{\partial v} = (31)4 + (54) = 178 = \frac{\partial z}{\partial v} \Big|_{(2,1,0)}$$

$$\frac{\partial z}{\partial w} = z_x \frac{\partial x}{\partial w} + z_y \frac{\partial y}{\partial w} = (31)(0) + (54)1 = 54 = \frac{\partial z}{\partial w} \Big|_{(2,1,0)}$$

Remark: there are other ways to organize  
 this calculation. But, should get same answers.

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§15.5#53] If  $z = f(x, y)$  where  $x = r \cos \theta$  and  $y = r \sin \theta$  show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r} \frac{\partial z}{\partial r} \quad (*)$$

This problem requires some persistence,

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = \boxed{z_x \cos \theta + z_y \sin \theta = z_r}$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r} [z_x \cos \theta + z_y \sin \theta]$$

$$= \frac{\partial z_x}{\partial r} \cos \theta + \frac{\partial z_y}{\partial r} \sin \theta$$

$$= \left( \frac{\partial z_x}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z_x}{\partial y} \frac{\partial y}{\partial r} \right) \cos \theta + \left( \frac{\partial z_y}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z_y}{\partial y} \frac{\partial y}{\partial r} \right) \sin \theta$$

$$\text{Eq. I} - \boxed{z_{rr} = z_{xx} \cos^2 \theta + z_{xy} \sin \theta \cos \theta + z_{yx} \cos \theta \sin \theta + z_{yy} \sin^2 \theta}$$

Now you need to calculate  $z_{\theta\theta}$  by a very similar calculation. When you put these into the RHS of (\*) things will cancel leaving just  $z_{xx} + z_{yy}$  as desired. Let's continue,

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} = -z_x r \sin \theta + z_y r \cos \theta$$

$$\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial}{\partial \theta} [-r z_x \sin \theta + r z_y \cos \theta]$$

$$= r \frac{\partial}{\partial \theta} [-z_x \sin \theta + z_y \cos \theta]$$

$$= r \left[ -\frac{\partial z_x}{\partial \theta} \sin \theta - z_x \cos \theta + \frac{\partial z_y}{\partial \theta} \cos \theta - z_y \sin \theta \right]$$

$$= r \left[ -\sin \theta \left( \frac{\partial z_x}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z_x}{\partial y} \frac{\partial y}{\partial \theta} \right) - z_x \cos \theta + \cos \theta \left( \frac{\partial z_y}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z_y}{\partial y} \frac{\partial y}{\partial \theta} \right) - z_y \sin \theta \right]$$

$$\text{Eq. II} - \boxed{z_{\theta\theta} = r \left[ -r \sin^2 \theta z_{xx} - r \sin \theta \cos \theta z_{xy} - z_x \cos \theta - r \sin \theta \cos \theta z_{yx} + r \cos^2 \theta z_{yy} - z_y \sin \theta \right]}$$

§15.5#53 Continued

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Now use Eq's (I), (II) and (III) together,

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = \leftarrow$$

$$\begin{aligned} \rightarrow &= \cos^2 \theta \frac{\partial^2 \phi}{\partial x^2} + \sin \theta \cos \theta \frac{\partial^2 \phi}{\partial x \partial y} + \sin \theta \cos \theta \frac{\partial^2 \phi}{\partial y \partial x} + \sin^2 \theta \frac{\partial^2 \phi}{\partial y^2} + \\ &+ \frac{1}{r^2} \left\{ r^2 \sin^2 \theta \frac{\partial^2 \phi}{\partial x^2} - r^2 \sin \theta \cos \theta \frac{\partial^2 \phi}{\partial x \partial y} - r^2 \sin \theta \cos \theta \frac{\partial^2 \phi}{\partial y \partial x} + r^2 \cos^2 \theta \frac{\partial^2 \phi}{\partial y^2} \right. \\ &\quad \left. - r \frac{\partial \phi}{\partial x} \cos \theta - r \frac{\partial \phi}{\partial y} \sin \theta \right\} + \frac{1}{r} \left\{ \frac{\partial \phi}{\partial x} \cos \theta + \frac{\partial \phi}{\partial y} \sin \theta \right\} \end{aligned}$$

$$\begin{aligned} &= (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2 \phi}{\partial x^2} + (\sin^2 \theta + \cos^2 \theta) \frac{\partial^2 \phi}{\partial y^2} \\ &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \end{aligned}$$

Thus we find,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r}$$

This is interesting since Laplace's Eq<sup>n</sup> is  $\nabla^2 \phi = 0$  and in two dimensions that reads

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

our calculation here reveals Laplace's Eq<sup>n</sup> in polar coordinates is written as

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = 0$$

Where  $\phi$  = potential energy per unit charge aka "the electric potential".

$$\nabla^2 \phi = -\rho/\epsilon_0 \text{ Poisson's Eq<sup>n</sup>}$$

When  $\rho = 0$  get Laplace's Eq<sup>n</sup>. Note  $\phi = \theta$  is a sol<sup>n</sup>. However,  $\phi = r$  is not. Just tinkering.