Ma 241-006 - §8.5 - selected sol^{n}'s

(#4) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ find the IOC and the radius of convergence R.

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{n+2} \cdot \frac{(n+1)}{(-1)^n x^n} \right|$$

$$= \lim_{n\to\infty} \left| -x \left(\frac{n+1}{n+2}\right) \right|$$

$$= |-x| \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right) = |x|$$

Ratio test says that series converges if $L = |x| < 1$. Now lets worry about the endpoints, the ratio test is inconclusive if $x = \pm 1$ so use other tests to check those values.

$x = 1$) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ is alternating series with $b_n = \frac{1}{n+1}$ this is clearly positive, decreasing and $\frac{1}{n+1} \to 0$ as $n \to \infty$ thus the series converges by A. S. T.

$x = -1$) $\sum_{n=0}^{\infty} (-1)^n (-1)^n \frac{1}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

this is the harmonic series, it diverges by the p-series test with $p = 1$.

Thus in total let us collect our work,

$$\boxed{IOC = (-1, 1] \quad \text{note } R = 1}$$

Remark: if we don't worry about the endpoints then all we need is to apply the ratio test. A sneaky way to phrase this is to ask you to find the "open interval of convergence" that would mean find the largest open set the power series converges on. That means
$(a+R, a-R)$ or $(-\infty, \infty)$

$\boxed{\begin{matrix}\S 8.5\\ \#6\end{matrix}}$ Find the largest open I.O.C for $\sum_{n=1}^{\infty} \sqrt{n}\, x^n$

Again apply ratio test.

$$L = \lim_{n\to\infty} \left| \frac{\sqrt{n+1}}{\sqrt{n}} \frac{x^{n+1}}{x^n} \right| = |x| \lim_{n\to\infty} \left(\sqrt{\frac{n+1}{n}} \right)$$

$$= |x| \lim_{n\to\infty} \left(\sqrt{1 + \underbrace{\tfrac{1}{n}}_{\to 0}} \right) = |x|.$$

Then the series converges by Ratio test when $L = |x| < 1$ thus the $\boxed{\text{largest open I.O.C} = (-1, 1) \text{ and } R = 1}$

(Not worrying about endpts. ½'s the work. btw. here both endpts diverge by n^{th} term test.)

$\boxed{\begin{matrix}\S 8.5\\ \#8\end{matrix}}$ $\sum_{n=1}^{\infty} \frac{x^n}{n 3^n}$ find open I.O.C. (ignore endpts.) and R.

$$L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)3^{n+1}} \frac{n 3^n}{x^n} \right|$$

$$= |x| \lim_{n\to\infty} \left| \frac{n}{3(n+1)} \right|$$

$$= \frac{1}{3}|x| < 1 \quad \Rightarrow \quad |x| < 3 \quad \left(\text{for the series to converge}\right)$$

By ratio test the series converges for $|x| < 3$. That is the $\boxed{\text{open I.O.C. is } (-3, 3) \text{ and } R = 3}$

$\boxed{\begin{matrix}\S 8.5\\ \#13\end{matrix}}$ $\sum_{n=1}^{\infty} (-1)^n \frac{(x+2)}{n 2^n}$ find the open I.O.C. and R.

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} (x+2)^{n+1}}{(n+1)\, 2^{n+1}} \frac{n\, 2^n}{(-1)^n (x+2)^n} \right|$$

$$= \lim_{n\to\infty} \left| -(x+2) \frac{n}{2(n+1)} \right|$$

$$= |x+2| \lim_{n\to\infty} \left(\frac{n}{2(n+1)} \right) = \frac{1}{2}|x+2|$$

Here $L < 1 \Rightarrow \frac{1}{2}|x+2| < 1 \Rightarrow |x+2| < 2$

$\Rightarrow -2 < x + 2 < 2$

$\Rightarrow -4 < x < 0$

$\Rightarrow \boxed{\text{open I.O.C} = (-4, 0) \;\&\; R = 2 \quad \text{By ratio test}}$

{I'd like you to think about the endpts.}

$\boxed{\S 8.5 \ \#15}$ $\sum_{n=1}^{\infty} \frac{n}{b^n}(x-a)^n$ assume $b>0$ find I.O.C. and R.

$$L = \lim_{n\to\infty}\left|\frac{n+1}{b^{n+1}}(x-a)^{n+1}\,\frac{b^n}{n(x-a)^n}\right|$$
$$= \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)\frac{1}{b}(x-a)\right|$$
$$= \frac{1}{b}|x-a| \lim_{n\to\infty}\left|\frac{n+1}{n}\right| = \frac{1}{b}|x-a|$$

Now if $L = \frac{1}{b}|x-a| < 1$ the series converges by ratio test. Equivalently:

$$L<1 \Rightarrow |x-a|<b \Rightarrow -b < x-a < b$$
$$\Rightarrow a-b < x < a+b$$
$$\Rightarrow x \in (a-b, a+b)$$

Clearly $R = b$ here, but what about the endpts?

$\underline{x = a-b}$ $\sum_{n=1}^{\infty}\frac{n}{b^n}(a-b-a)^n = \sum_{n=1}^{\infty}\frac{n}{b^n}(-b)^n = \sum_{n=1}^{\infty}(-1)^n n$

diverges by n^{th} term test as $(-1)^n n \to \infty$ as $n\to\infty$.

$\underline{x = a+b}$ $\sum_{n=1}^{\infty}\frac{n}{b^n}(a+b-a)^n = \sum_{n=1}^{\infty} n$

diverges by n^{th} term test as $n \to \infty$ as $n\to\infty$.

Thus neither endpt. enters the I.O.C., that is $\boxed{\text{I.O.C.} = (a-b, a+b)}$

$\boxed{\S 8.5 \ \#17}$ $L = \lim_{n\to\infty}\left|\frac{(n+1)!\,(2x-1)^n}{n!\,(2x-1)^n}\right| = |2x-1| \lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\right)$

$$= |2x-1| \lim_{n\to\infty}\left(\frac{(n+1)\cancel{n}\cancel{(n-1)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{n}\cancel{(n-1)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}\right)$$
$$= |2x-1| \lim_{n\to\infty}(n+1)$$

$(x \neq \frac{1}{2}) \longrightarrow \ = \infty > 1$ thus the series almost always diverges. When $x = 1/2$ the series is identically zero. Thus

I.O.C. $= \{\frac{1}{2}\}$

§8.6 #4

$f(x) = \dfrac{3}{1-x^4}$ geometric with $a = 3$, $r = x^4$

$= 3 + 3x^4 + 3x^8 + 3x^{12} + \cdots$ (for $|x^4| < 1$)

$f(x) = \sum_{n=0}^{\infty} 3x^{4n}$ for I.O.C $= (-1,1)$ by geometric series since $x \in (-1,1)$ insure that $|r| < 1$ and we know $|r| \geq 1$ causes the geometric series to diverge. No need for ratio test here, we get the I.O.C. almost for free, geometric series are the best.

§8.6 #6

$f(x) = \dfrac{1}{1+9x^2} = 1 - 9x^2 + (-9x^2)^2 + (-9x^2)^3 + \cdots$ $|-9x^2| < 1$

Cleaning it up a little, $\dfrac{1}{1+9x^2} = \sum_{n=0}^{\infty} (-1)^n (9x^2)^n$ when

$|-9x^2| < 1 \Rightarrow |9x^2| < 1 \Rightarrow |x^2| < \frac{1}{9} \Rightarrow |x| < \frac{1}{3}$

The I.O.C here is $(-1/3, 1/3)$.

§8.6 #8

$f(x) = \dfrac{x}{1+4x} = \sum_{n=0}^{\infty} x(-4x)^n = \sum_{n=0}^{\infty} (-1)^n 4^n x^{n+1}$

and this holds for $|r| = |-4x| < 1 \Rightarrow |x| < \frac{1}{4} \Rightarrow \boxed{\left(\frac{-1}{4}, \frac{1}{4}\right) \text{ I.O.C.}}$

§8.6 #10

$\dfrac{x^2}{a^3 - x^3} = \dfrac{x^2}{a^3(1 - x^3/a^3)}$, assuming $a \neq 0$

Geometric series with "a" $= x^2/a^3$ and $r = (x/a)^3$ (Sorry "a" and a are not the same here, the book made me do it...) Notation aside, we find

$$\boxed{\frac{x^2}{a^3 - x^3} = \sum_{n=0}^{\infty} \frac{x^2}{a^3}\left(\left(\frac{x}{a}\right)^3\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{a}\right)^{3n+3} x^{3n+2} = \frac{x^2}{a^3} + \frac{x^5}{a^6} + \frac{x^8}{a^9} + \cdots}$$

The I.O.C. contains x such that $|(x/a)^3| < 1$

$|x^3| < |a|^3$

$|x| < a$

$\boxed{\text{I.O.C} = (-a, a)}$