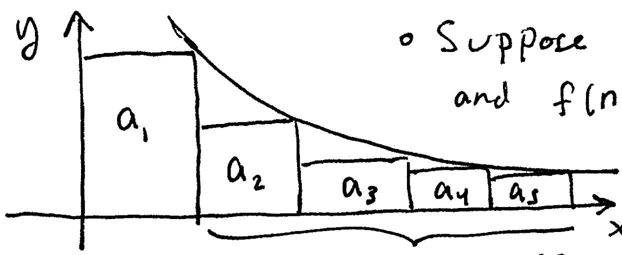
Examples of Integral Test and Estimation of Series

- upto this point we only had
  - direct calculation of S_n to decide conv/div. of series
  - geometric series $\underbrace{a + ar + ar^2 + \cdots = \frac{a}{1-r} \text{ if } |r| < 1}$
  - n^{th} term test

  b.t.w. if $|r| \geq 1$ then the geometric series diverges by the n^{th} term test.

- In this set of examples we'll introduce a few new tools to decide conv/div. of $\sum a_n$
  - integral test
  - p-series

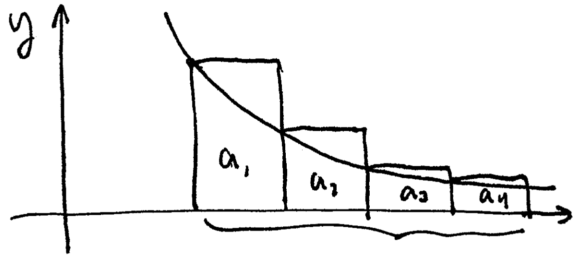
Also, we study an associated estimation result which tells us how close $S_n = a_1 + a_2 + \cdots + a_n$ is to $a_1 + a_2 + \cdots$ (when the series is convergent). These pictures tell you why the $\int$ test works.



- Suppose $a_i \geq 0$ and $f(n) = a_n$ is continuous fnct.

$$\underset{\text{under } y=f(x)}{\text{area}} = \int_1^\infty f(x)\,dx \geq a_2 + a_3 + \cdots$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^\infty f(x)\,dx \quad \text{(I.)}$$



$$\text{area under } y=f(x) = \int_1^\infty f(x)\,dx \leq a_1 + a_2 + \cdots \quad \text{(II.)}$$

Thus from (I) & (II), $\int_1^\infty f(x)\,dx \leq a_1 + a_2 + \cdots \leq a_1 + \int_1^\infty f(x)\,dx.$

the pictures on the last page motivate,

Th$^{\underline{m}}$ (Integral Test): Suppose f is <u>continuous</u>, <u>positive</u> and <u>decreasing</u> on $[1, \infty)$ and let $a_n = f(n)$ for $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} a_n$ is convergent iff $\int_1^{\infty} f(x)\,dx$ is convergent. Thus, $\sum_{n=1}^{\infty} a_n$ is divergent iff $\int_1^{\infty} f(x)\,dx$ is divergent.

1.) <u>HARMONIC SERIES</u>: $\sum_{n=1}^{\infty} \frac{1}{n}$ study $f(x) = \frac{1}{x}$.

Notice $f(x) = \frac{1}{x} > 0$ for $x \in [1, \infty)$ and f is continuous. Note $\frac{df}{dx} = \frac{-1}{x^2} < 0 \Rightarrow f$ decreasing on $[1, \infty)$.

Thus study,

$$\int_1^{\infty} \frac{dx}{x} = \lim_{b \to \infty}\left(\int_1^{b} \frac{dx}{x}\right)$$

$$= \lim_{b \to \infty}\left(\ln(b) - \ln(1)\right)$$

$$= \boxed{\infty}$$

Thus $\sum_{n=1}^{\infty} \frac{1}{n}$ <u>diverges</u> by the Integral Test.

<u>Remark</u>: the "answer" in 1. is the entire discussion. If you say $\boxed{\text{diverges by integral test}}$ but don't support the hypotheses of the test with <u>sentences</u> then I should not give full-credit.

2.) <u>P-SERIES</u>: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{convergent for } p>1 \\ \text{divergent for } p \le 1 \end{cases}$

To prove the claim above we can use n^{th} term test for $p \le 0$ in which case $\frac{1}{n^p} \not\to 0$ as $n \to \infty$.

Thus assume $p > 0$ and notice $\frac{1}{n^p} = f(n)$ for

$$\underbrace{f(x) = \frac{1}{x^p}}_{\text{continuous and positive on } [1,\infty)}$$

Also, $\frac{df}{dx} = \frac{-p}{x^{p+1}} < 0 \Rightarrow f$ decreasing on $[1,\infty)$.

We already proved $p=1$ is divergent (HARMONIC SERIES)

So assume $p>0$ and $p \ne 1$ and calculate,

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{b\to\infty}\left[\int_1^b x^{-p}\,dx\right]$$

$$= \underbrace{\lim_{b\to\infty}\left[\frac{b^{-p+1}}{1-p} - \frac{1}{1-p}\right]}_{\text{diverges to } \infty \text{ for } 1-p>0} \overset{\text{for } 1-p<0}{=} \frac{1}{p-1}$$

Thus $\int_1^{\infty} \frac{dx}{x^p}$ converges for $p>1$ and diverges for $0<p<1$.

The P-SERIES result follows by the integral test.

<u>Remark</u>: $\int_1^{\infty} \frac{dx}{x^2} = \frac{1}{2-1} = 1$ yet it can be shown by more advanced methods that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

It is important to realize the $\int$-test speaks to CONV/DIV not to the precise value of the series... <u>BUT</u> we do have

$$\int_{n+1}^{\infty} f(x)\,dx \le \underbrace{S - S_n}_{R_n} \le \int_n^{\infty} f(x)\,dx$$

R_n ← tail or remainder of series S.

3.) estimate error possible in <u>truncating</u> $p=2$ series after 3 - terms. If $S = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$ then $|S - S_3| \le \varepsilon_3$? (find ε_3 ← error bound)

<u>Notation</u>: $S = \sum_{n=1}^{\infty} a_n$ and $S = \underbrace{\sum_{k=1}^{n} a_k}_{S_n} + \underbrace{\sum_{k=n+1}^{\infty} a_k}_{R_n}$

$S - S_n = R_n$

and by pictures similar to those shown earlier,

$$\underbrace{\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx}_{\text{for } f \text{ satisfying } \int\text{-test criteria.}}$$

In $p=2$ case,

$$\int_{n+1}^{\infty} \frac{dx}{x^2} = \lim_{b \to \infty} \left[\int_{n+1}^{b} \frac{dx}{x^2} \right]$$

$$= \lim_{b \to \infty} \left[\frac{-1}{b} + \frac{1}{n+1} \right]$$

$$= \frac{1}{n+1}$$

Likewise, $\int_{n}^{\infty} \frac{dx}{x^2} = \frac{1}{n}$ thus $\frac{1}{n+1} \le R_n \le \frac{1}{n}$

For $n = 3$, $\frac{1}{4} \le R_3 \le \frac{1}{3}$ $\Rightarrow$ choose $\boxed{\varepsilon_3 = \frac{1}{3}}$.

Thus $\underbrace{1 + \frac{1}{4} + \frac{1}{9}}_{\approx 1.3611} \cong \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\frac{\pi^2}{6} \cong 1.6449}$ to within $\varepsilon_3 = \frac{1}{3}$

$\Rightarrow$ $\underbrace{R_3 \approx 0.2838}_{\text{between } 1/4 \text{ and } 1/3 \text{ as expected.}}$

<u>Remark</u>: we didn't need to know but it's fun to see it works.

4.) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$ conv or diverge ?

Let $f(x) = x^2 e^{-x^3}$ and notice $f(x) > 0$ and f is continuous on $[1, \infty)$. Also, differentiate,

$$\frac{df}{dx} = 2x e^{-x^3} + x^2(-3x^2)e^{-x^3}$$

$$= \underbrace{x e^{-x^3}}_{\substack{\text{positive}\\ \text{on}\\ [1,\infty)}} \underbrace{(2 - 3x^3)}_{\substack{\text{*negative}\\ \text{on}\\ [1,\infty)}} < 0 \quad \therefore f \text{ decreasing.}$$

} I see this from the sign-chart below

$$* \quad 2 - x^3(3) = 0 \Rightarrow x^3 = \frac{2}{3} \Rightarrow x = \sqrt[3]{\frac{2}{3}} < 1$$

$+ + + + + + \; | \; - - - - - \longrightarrow 2 - 3x^3$

$\sqrt[3]{2/3}$

Thus f is decreasing on $[1, \infty)$ (actually more, but this is all we need)

Note $\displaystyle\int x^2 e^{-x^3}\,dx = \frac{-1}{3} e^{-x^3} + C$ by $u = -x^3$ subst.

Hence, $\displaystyle\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{b \to \infty} \left[\frac{-1}{3} e^{-b^3} + \frac{1}{3} e^{-1} \right] = \frac{1}{3e}$

Therefore, $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges by integral test.

5.) Find sum of $\underbrace{\sum_{n=1}^{\infty} n e^{-2n}}$ correct to 4 decimal places.

$$S = \_\_ . \_\_\_\_\_ \boxed{}$$

uncertain digit.

GOAL: $|R_n| \leq 10^{-5}$

$\varepsilon_n = 10^{-5}$ will certainly suffice.

$$\int \underbrace{x}_{u} \underbrace{e^{-2x} dx}_{dV} = -\frac{1}{2} x e^{-2x} + \frac{1}{2}\int e^{-2x} dx$$

$$= \underline{-\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + C}.$$

You can check $f(x) = xe^{-2x}$ is positive, continuous and decreasing on $[1, \infty)$ (actually it suffices for this to be true for some $M > 0$ and $[M, \infty)$, what happens for $1 < x < M$ matters not as long as f is finite)

$$\int_n^{\infty} xe^{-2x} dx = \lim_{b \to \infty} \Big[\underbrace{-\frac{1}{2} b e^{-2b}}_{\substack{\text{apply} \\ \text{L-Hop to} \\ \text{see} \to 0}} - \frac{1}{4} \underbrace{e^{-2b}}_{\to 0} + \underbrace{\frac{1}{2} n e^{-2n} + \frac{1}{4} e^{-2n}}_{\frac{1}{4}(2n+1)e^{-2n}} \Big]$$

Thus, by $\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$ we find estimate,

$$\frac{1}{4}(2(n+1)+1)e^{-2(n+1)} \leq R_n \leq \underbrace{\frac{1}{4}(2n+1)e^{-2n}}_{\text{we need this} \leq 10^{-5}} = \varepsilon$$

use calculator to punch in $n = 3, 4, \ldots$

$n = 3$ gives $\varepsilon \approx 4.34 \times 10^{-3}$

$n = 4$ gives $\varepsilon \approx 7.54 \times 10^{-4}$

$n = 5$ gives $\varepsilon \approx 1.25 \times 10^{-4}$

$n = 6$ gives $\varepsilon \approx 1.99 \times 10^{-5}$

good enough, we can use S_5 for the desired accuracy.

5.) we showed $\left| \sum_{n=1}^{\infty} ne^{-2n} - \sum_{n=1}^{5} ne^{-2n} \right| \leq 2\times10^{-5}$

Thus calculate,

$$S_5 = \sum_{n=1}^{5} ne^{-2n}$$

$$= \frac{1}{e^2} + \frac{2}{e^4} + \frac{3}{e^6} + \frac{4}{e^8} + \frac{5}{e^{10}}$$

$$= \frac{5 + 4e^2 + 3e^4 + 2e^6 + e^8}{e^{10}}$$

$$\cong \underbrace{\boxed{0.1809}}_{\text{certain.}}72$$

oops! forgot to round, $\boxed{0.1810}$

B.t.w. wolframalpha.com makes it easy to make such calculations. You can check,

$$\sum_{n=1}^{10} ne^{-2n} \cong \underline{0.1810}154..$$

In fact, in this case wolframalpha.com claims

$$\sum_{n=1}^{\infty} ne^{-2n} = \frac{e^2}{(e^2-1)^2}$$

this probably means <u>complex analysis</u> (MATH 331) allows direct calculation via some trick ...