INTERESTING INTEGRALS

- we've learned IBP, trig. integrals, implicit trig or hyperbolic substitutions and partial fractions. Now <u>combine</u> these,

1.) $$\int \sec\theta \, d\theta = \int \frac{d\theta}{\cos\theta}$$

(YES, $u = \sec\theta + \tan\theta$ yields $\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$ but we investigate another approach here.)

$$= \int \frac{\cos\theta \, d\theta}{\cos^2\theta}$$

$$= \int \frac{\cos\theta \, d\theta}{1 - \sin^2\theta}$$

$\boxed{u = \sin\theta, \quad du = \cos\theta \, d\theta}$

$$= \int \frac{du}{1-u^2} \quad :$$

$$\frac{1}{1-u^2} = \frac{A}{u-1} + \frac{B}{u+1}$$

$$1 = A(u+1) + B(u-1)$$

$\underline{u=1|}$ $\quad 1 = 2A \quad \therefore \quad A = 1/2$

$\underline{u=-1|}$ $\quad 1 = -2B \quad \therefore \quad B = -1/2$

$$= \frac{1}{2}\int \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du$$

$$= \frac{1}{2}\Big(\ln|u-1| - \ln|u+1|\Big) + C$$

$$= \boxed{\frac{1}{2}\ln|\sin\theta - 1| - \frac{1}{2}\ln|\sin\theta + 1| + C} - *$$

2.) $$\int \underbrace{\ln(x^2-3x+2)}_{u}\,\underbrace{dx}_{dV} \overset{\text{IBP}}{=} x(x^2-3x+2) - \underbrace{\int \frac{x(2x-3)\,dx}{x^2-3x+2}}_{*}$$

Let's work on $*$ alone,

$$* = \int \frac{2x^2-3x}{x^2-3x+2}\,dx \quad : \quad x^2-3x+2 \,\overset{2}{\overline{\big)\,2x^2-3x}}$$

$$\frac{2x^2-6x+4}{\boxed{3x-4}}$$ remainder.

$$= \int \left(2 + \frac{3x-4}{x^2-3x+2}\right)dx$$

$$\frac{3x-4}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$3x-4 = A(x-2) + B(x-1)$$

$\underline{x=1|}$ $-1 = -A$ ∴ $\underline{A=1}$.

$\underline{x=2|}$ $\underline{2 = B}$.

$$= \int \left(2 + \frac{1}{x-1} + \frac{2}{x-2}\right)dx$$

$$= \underbrace{2x + \ln|x-1| + 2\ln|x-2| + C}_{*}$$

Consequently,

$$\int \ln(x^2-3x+2)\,dx = x^3-3x^2+2x-\left(2x+\ln|x-1|+2\ln|x-2|\right)+C$$

$$= \boxed{x^3-3x^2-\ln|x-1|-2\ln|x-2|+C}$$

3.) $$\int \frac{e^x\,dx}{4+e^{2x}} = \int \frac{du}{4+u^2} \quad : \boxed{u = e^x,\ du = e^x dx}$$

$$= \int \frac{2\sec^2\theta\,d\theta}{4\sec^2\theta} \qquad \boxed{\begin{array}{l} u = 2\tan\theta,\ du = 2\sec^2\theta\,d\theta \\ u^2+4 = 4\sec^2\theta \end{array}}$$

$$= \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \boxed{\frac{1}{2}\tan^{-1}\left(\frac{e^x}{2}\right) + C}$$

4.) $$\int\sqrt{\frac{3+x}{3-x}}\,dx = \int\sqrt{\frac{9-x^2}{(3-x)^2}}\,dx$$: multiplied by $\frac{3-x}{3-x}$

$$= \int \frac{\sqrt{9-x^2}}{3-x}\,dx$$: assuming $3-x>0$.

$$= \int \frac{9\cos^2\theta\,d\theta}{3-3\sin\theta}$$

$x = 3\sin\theta$
$9-x^2 = 9\cos^2\theta$
$dx = 3\cos\theta\,d\theta$

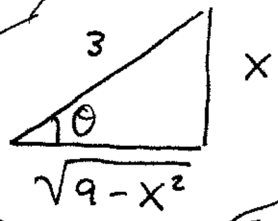
$$= \int 3\left(\frac{1-\sin^2\theta}{1-\sin\theta}\right)d\theta$$

$$= 3\int \frac{(1+\sin\theta)\cancel{(1-\sin\theta)}\,d\theta}{\cancel{1-\sin\theta}}$$

$$= 3\int (1+\sin\theta)\,d\theta$$

$$= 3\theta + 3\cos\theta + C$$

$\sin\theta = \frac{x}{3}$



$$= 3\sin^{-1}\left(\frac{x}{3}\right) + 3\left(\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \boxed{3\sin^{-1}\left(\frac{x}{3}\right) + \sqrt{9-x^2} + C}$$

5.) $$\int \cos(\sqrt{x})\,dx = \int \overbrace{\cos(w)\cdot \underset{u}{\underline{2w}}\,dw}^{dv}$$

plain old substitution

$W = \sqrt{x}$
$W^2 = x$
$2W\,dW = dx$

$$= 2w\sin(w) - \int 2\sin(w)\,dw$$

$$= 2w\sin(w) + 2\cos(w) + C$$

$$= \boxed{2\sqrt{x}\,\sin\sqrt{x} + 2\cos\sqrt{x} + C}$$