LECTURE 10 : CLASSICAL FIELD THEORY

The problem of classical field theory is to explain the dynamics of fields on spacetime. Generically $\Phi^i(x^\mu)$ amounts to ∞ly many variables. Once again a variational principle is used to set-up the calculus for classical fields,

$$L = \int d^3x \, \mathcal{L}(\Phi^i, \partial_\mu \Phi^i)$$

Here $\mathcal{L}$ is the "Lagrange density" which depends on fields and their derivatives. Often the Lagrange density is called the Lagrangian in this context. The action is given by:

$$S = \int dt \, L = \int d^4x \, \mathcal{L}(\Phi^i, \partial_\mu \Phi^i)$$

We vary the fields and suppose $\delta S = 0 \Rightarrow$ Equations of Motion for fields.

$$\mathcal{L}(\Phi^i, \partial_\mu \Phi^i) + \delta\mathcal{L} = \mathcal{L}(\Phi^i + \delta\Phi^i, \partial_\mu \Phi^i + \partial_\mu \delta\Phi^i) \qquad \delta(\partial_\mu \Phi^i) = \partial_\mu(\delta\Phi^i)$$

$$= \mathcal{L}(\Phi^i, \partial_\mu \Phi^i) + \frac{\partial \mathcal{L}}{\partial \Phi^i}\delta\Phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\partial_\mu \delta\Phi^i$$

when we vary S we find,

$$\delta S = \int d^4x \, \delta\mathcal{L} = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \Phi^i}\delta\Phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\partial_\mu \delta\Phi^i \right)$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \Phi^i}\delta\Phi^i + \cancel{\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\delta\Phi^i \right]} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)} \right) \delta\Phi^i \right) \xrightarrow{} 0$$

By 4-D Stokes' Thm... usually...

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)} \right) \right) \delta\Phi^i$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\right) = 0 \qquad \text{Euler-Lagrange Eq}^{\underline{n}}\text{s for fields on spacetime}}$$

<u>Example</u>: real scalar field (e.g. the neutral π-meson)

kinetic energy $\frac{1}{2}\dot{\phi}^2$

gradient energy $\frac{1}{2}(\nabla\phi)^2$

potential energy $V(\phi)$

The potential energy is a scalar function which is Lorentz invariant. However, the kinetic and gradient terms must be combined,

$$-\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2$$

Then in total,

$$\mathcal{L} = \frac{-1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$$

What are the eq$^{\underline{n}}$s of motion here?

$$\frac{\partial\mathcal{L}}{\partial\phi} = -\frac{dV}{d\phi} \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} = \frac{-1}{2}\eta^{\mu\nu}\left(\overbrace{\frac{\partial(\partial_\mu\phi)}{\partial(\partial_\alpha\phi)}}^{\delta_{\mu\alpha}}\partial_\nu\phi + \partial_\mu\phi\overbrace{\frac{\partial(\partial_\nu\phi)}{\partial(\partial_\alpha\phi)}}^{\delta_{\nu\alpha}}\right)$$

$$= \frac{-1}{2}\eta^{\mu\nu}\left(\delta_{\mu\alpha}\partial_\nu\phi + \underline{\delta_{\nu\alpha}\partial_\mu\phi}\right)$$

$$= -\eta^{\mu\nu}\delta_{\mu\alpha}\partial_\nu\phi$$

relable $\mu \to \nu$, $\nu \to \mu$ and use $\eta^{\mu\nu} = \eta^{\nu\mu}$ to collapse 2$^{\text{nd}}$ term.

$$= -\eta^{\alpha\nu}\partial_\nu\phi = -\partial^\alpha\phi$$

continued

Euler Lagrange eq$^{\underline{ns}}$ for $\mathcal{L} = \frac{-1}{2}\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) - V(\phi)$

are $\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$ thus,

$$-\frac{dV}{d\phi} - \partial_\mu\left(-\eta^{\mu\nu}\partial_\nu \phi\right) = 0$$

$$\partial_\mu \partial^\mu \phi - \frac{dV}{d\phi} = 0 \quad \text{a.k.a.} \quad \boxed{\Box\phi - \frac{dV}{d\phi} = 0}$$

$\boxed{\text{Def}^{\underline{n}}/ \; \Box = \partial_\mu \partial^\mu = -\frac{d^2}{dt^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = -\partial_t^2 + \nabla^2 \text{ the d'Alembertian}}$

If we're given $V(\phi) = \frac{1}{2}m^2\phi^2$ (simple harmonic oscillator)

$$\boxed{\Box\phi - m^2\phi = 0 \quad \text{Klein-Gordon equation}}$$

ELECTROMAGNETISM FROM LAGRANGIAN FORMALISM (§1.10 CARROLL)

Introduce vector potential $(A_\mu) = (\Phi, \vec{A})$ where $\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \nabla \times \vec{A}$

We can recover $\vec{E}$ and $\vec{B}$ inside the field strength $F_{\mu\nu}$ and you can show

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu} \quad (*)$$

<u>Gauge freedom</u>: replace $A_\mu \longmapsto A_\mu + \partial_\mu \lambda$ where λ is function on spacetime. Notice $F_{\mu\nu} \longmapsto F_{\mu\nu}$ since $\partial_\mu \partial_\nu \lambda - \partial_\nu \partial_\mu \lambda = 0$. Thus $F_{\mu\nu}$ is <u>gauge</u> <u>invariant</u>

- CARROLL POINTS OUT $\partial_{[\mu} F_{\nu\sigma]} = 0$ automatically if we use $(*)$ to define $F_{\mu\nu}$, or at least the homogeneous Maxwell's Eq's come for free in the $(*)$ formalism.

$$\partial_{[\mu} \partial_\nu A_{\sigma]} - \partial_{[\mu} \partial_\sigma A_{\nu]} = 0 \quad \text{since } \partial_\mu \partial_\nu = \partial_\nu \partial_\mu \text{ etc...}$$

Let me take a moment to explain these claims with <u>differential</u> <u>forms</u>

Electromagnetism with differential forms

$A^\mu = (\Phi, \vec{A})$ Griffiths States

$$A = A_\mu dx^\mu = -\Phi\, dt + \omega_{\vec{A}} \quad \text{where} \quad \underbrace{\omega_{\vec{A}} = A_1\, dx + A_2\, dy + A_3\, dz}_{\text{work-form correspondence of vector field } \vec{A} \text{ to one-form } \omega_{\vec{A}}}$$

$$dA = -d\Phi \wedge dt + d\omega_{\vec{A}}$$

$$= \left(\frac{-\partial\Phi}{\partial t} dt - \omega_{\nabla\phi}\right) \wedge dt + - \omega_{\frac{\partial \vec{A}}{\partial t}} \wedge dt + \Phi_{\nabla \times \vec{A}}$$

$$= \left(\omega_{-\nabla\phi} - \frac{\partial \vec{A}}{\partial t}\right) \wedge dt + \Phi_{\nabla \times \vec{A}}$$

$$= \underbrace{\omega_{\vec{E}} \wedge dt + \Phi_{\vec{B}}}_{\text{compare to } [F_{\mu\nu}] \text{ from page (2) of lecture 8}} = F$$

$$\underbrace{\Phi_{\langle a,b,c\rangle} = a\, dy \wedge dz + b\, dz \wedge dx + c\, dx \wedge dy}_{\text{flux-form correl. of a vector field } \langle a,b,c\rangle \text{ to the two-form } \Phi\langle a,b,c\rangle}$$

Define $F = dA$ then $dF = d(dA) = 0$ gives the homogeneous Maxwell Eq's,

$$dF = d\omega_{\vec{E}} \wedge dt + d\Phi_{\vec{B}} = \Phi_{\nabla \times \vec{E}} \wedge dt + \Phi_{\frac{\partial \vec{B}}{\partial t}} \wedge dt + (\nabla \cdot \vec{B})\, dx \wedge dy \wedge dz$$

$$\Rightarrow \quad \underline{\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0} \quad \text{and} \quad \underline{\nabla \cdot \vec{B} = 0}.$$

E & M with differential forms continued

Gauge transformation $A \longmapsto A + d\lambda$

$$F \longmapsto d(A + d\lambda) = dA + d^2\lambda = F$$

Hodge Duality: swaps p for $n-p$ forms

$$* dt = \pm dx \wedge dy \wedge dz$$
$$* dx \wedge dy = \pm dt \wedge dz$$
$$* dx \wedge dy \wedge dz = \pm dt$$

I'm not being careful here... sorry.

Summary:
Maxwell's Eq$^{\text{n}}$'s are given by $F = dA$ where we observe
$$dF = 0$$
$$d * F = * J$$

$$* \left(\omega_{\vec{E}} \wedge dt + \Phi_{\vec{B}} \right) = \omega_{\vec{B}} \wedge dt - \Phi_{\vec{E}}$$

$$d(*F) = d\omega_{\vec{B}} \wedge dt - d\Phi_{\vec{E}}$$

$$= \Phi_{\nabla \times \vec{B}} \wedge dt + \cancel{dt \wedge \omega_{\frac{\partial \vec{B}}{\partial t}} \wedge dt}^{\,0} - dt \wedge \Phi_{\frac{\partial \vec{E}}{\partial t}} - (\nabla \cdot \vec{E}) dx \wedge dy \wedge dz$$

$$= \left(\Phi_{\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t}} \right) \wedge dt - (\nabla \cdot \vec{E}) dx \wedge dy \wedge dz$$

$$= * J \quad \text{where} \quad \underbrace{J = \rho\, dt + \omega_{\vec{J}}}_{\text{current one-form}}$$

has $\underbrace{*J = -\rho\, dx \wedge dy \wedge dz + \Phi_{\vec{J}} \wedge dt}_{\text{Hodge dual of current one-form.}}$

E&M in Lagrangian Formalism

Ryder

$$\mathcal{L} = \frac{-1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = 0}$$

- * - Euler Lagrange Eq^{n} for A_ν

We seek to calculate the EL-eq^{n}'s for the above Lagrangian.

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{\partial}{\partial A_\nu}\left(A_\mu J^\mu\right) = \frac{\partial A_\mu}{\partial A_\nu} J^\mu = \delta^\mu_\nu J^\mu = J^\nu$$

We assume independence of A_μ and $\partial_\nu A_\mu$ in assuming the $\frac{-1}{4}F^2$ term contributes nothing to the above derivative.

$$F_{\mu\nu} F^{\mu\nu} = F_{\alpha\beta} F^{\alpha\beta} = \eta^{\alpha\rho}\eta^{\beta\sigma} F_{\alpha\beta} F_{\rho\sigma}$$

Observe that

$$\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = \frac{\partial}{\partial(\partial_\mu A_\nu)}\left[\partial_\alpha A_\beta - \partial_\beta A_\alpha\right] = \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}$$

Thus calculate,

$$\begin{aligned}
\frac{\partial(F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} &= \eta^{\alpha\rho}\eta^{\beta\sigma}\frac{\partial}{\partial(\partial_\mu A_\nu)}\left[F_{\alpha\beta}F_{\rho\sigma}\right] \\
&= \eta^{\alpha\rho}\eta^{\beta\sigma}\Big(\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right)F_{\rho\sigma} + F_{\alpha\beta}\left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right)\Big) \\
&= \eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma}F_{\rho\sigma} + \eta^{\alpha\mu}\eta^{\beta\nu}F_{\alpha\beta} - \eta^{\alpha\nu}\eta^{\beta\mu}F_{\alpha\beta} \\
&= F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu} \\
&= 4F^{\mu\nu}
\end{aligned}$$

Consequently, we have shown $\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}$ $\quad\therefore\quad \boxed{\partial_\mu F^{\nu\mu} = J^\nu}$

(same as Lecture 8)

- Energy-Momentum-Tensors for the examples in this Lecture: (from a mysterious procedure in Chpt. 4)

$$T^{\mu\nu}_{\substack{\text{scalar}\\\text{field}}} = \eta^{\mu\lambda}\eta^{\nu\sigma}\partial_\lambda\phi\,\partial_\sigma\phi - \eta^{\mu\nu}\left[\tfrac{1}{2}\eta^{\lambda\sigma}\partial_\lambda\phi\,\partial_\sigma\phi + V(\phi)\right]$$

$$T^{\mu\nu}_{EM} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}$$

- <u>SOME COMMENTS ON STANDARD MODEL OF PARTICLE PHYSICS</u>

  - GAUGE FIELDS
    - Strong force (gluons) } spin 1
    - Weak Force ($W^\pm$, Z^0 bosons) } spin 1
    - Electromagnetic force (photon) } spin 1
    - Gravity (gravitino) } spin 2
  - HIGGS FIELDS (Higg's Boson) } spin 0
  - FERMIONS
    - electrons } spin $\frac{1}{2}$
    - neutrinos } spin $\frac{1}{2}$
    - quarks } spin $\frac{1}{2}$

- All fields carry matrix rep. of S.M. symmetry group $SU(3)\times SU(2)\times U(1)$
- Also need to consider <u>SPIN</u> of the particles as given by <u>spinors</u>
- Quantum Mechanics, well QUANTUM FIELD THEORY, allows for interpretation of low-energy physics as an <u>effective field theory</u>
The high-energy processes "renormalize" the coupling constants...
in short, we find decoupling of physics at differing energy regimes...
- WHERE we're headed, GR has <u>metric tensor</u> as dynamic field of the theory.