(b) $\lim_{n\to\infty} \frac{3^n}{n!}$.

(c) $\lim_{n\to\infty} \frac{n!}{n^n}$.

(d) $\lim_{n\to\infty} \frac{n^2}{3^n}$. (*Hint:* see Exercise 1.3.4(c)).

2.1.7 Prove that if $\lim_{n\to\infty} a_n = \ell > 0$, then there exists $N \in \mathbb{N}$ such that $a_n > 0$ for all $n \geq N$.

2.1.8 ▶ Prove that if $\lim_{n\to\infty} a_n = \ell \neq 0$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Is the conclusion still true if $\ell = 0$?

2.1.9 Let $\{a_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = 3$. Use Definition 2.1.1 to prove the following

(a) $\lim_{n\to\infty} 3a_n - 7 = 2$;

(b) $\lim_{n\to\infty} \frac{a_n + 1}{a_n} = \frac{4}{3}$; (*Hint:* prove first that there is N such that $a_n > 1$ for$n \geq N$.)

2.1.10 Let $a_n \geq 0$ for all $n \in \mathbb{N}$. Prove that if $\lim_{n\to\infty} a_n = \ell$, then $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{\ell}$.

2.1.11 Prove that the sequence $\{a_n\}$ with $a_n = \sin(n\pi/2)$ is divergent.

2.1.12 ▶ Consider a sequence $\{a_n\}$.

(a) Prove that $\lim_{n\to\infty} a_n = \ell$ if and only if $\lim_{k\to\infty} a_{2k} = \ell$ and $\lim_{k\to\infty} a_{2k+1} = \ell$.

(b) Prove that $\lim_{n\to\infty} a_n = \ell$ if and only if $\lim_{k\to\infty} a_{3k} = \ell$, $\lim_{k\to\infty} a_{3k+1} = \ell$, and $\lim_{k\to\infty} a_{3k+2} = \ell$.

2.1.13 Given a sequence $\{a_n\}$, define a new sequence $\{b_n\}$ by

$$b_n = \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

(a) Prove that if $\lim_{n\to\infty} a_n = \ell$, then $\lim_{n\to\infty} b_n = \ell$.

(b) Find a counterexample to show that the converse does not hold in general.

2.2 LIMIT THEOREMS (LECTURE 10)

We now prove several theorems that facilitate the computation of limits of some sequences in terms of those of other simpler sequences.

Theorem 2.2.1 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let k be a real number. Suppose $\{a_n\}$ converges to a and $\{b_n\}$ converges to b. Then the sequences $\{a_n + b_n\}$, $\{ka_n\}$, and $\{a_nb_n\}$ converge and

(a) $\lim_{n\to\infty}(a_n + b_n) = a + b$;

(b) $\lim_{n\to\infty}(ka_n) = ka$;

(c) $\lim_{n\to\infty}(a_nb_n) = ab$;

(d) If in addition $b \neq 0$ and $b_n \neq 0$ for $n \in \mathbb{N}$, then $\left\{\frac{a_n}{b_n}\right\}$ converges and $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

Proof: (a) Fix any $\varepsilon > 0$. Since $\{a_n\}$ converges to a, there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\varepsilon}{2} \text{ for all } n \geq N_1.$$

Similarly, there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - b| < \frac{\varepsilon}{2} \text{ for all } n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. For any $n \geq N$, one has

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\lim_{n\to\infty}(a_n + b_n) = a + b$. This proves (a).

(b) If $k = 0$, then $ka = 0$ and $ka_n = 0$ for all n. The conclusion follows immediately. Suppose next that $k \neq 0$. Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $|a_n - a| < \frac{\varepsilon}{|k|}$ for $n \geq N$. Then for $n \geq N$, $|ka_n - ka| = |k||a_n - a| < \varepsilon$. It follows that $\lim_{n\to\infty}(ka_n) = ka$ as desired. This proves (b).

(c) Since $\{a_n\}$ is convergent, it follows from Theorem 2.1.7 that it is bounded. Thus, there exists $M > 0$ such that

$$|a_n| \leq M \text{ for all } n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$, we have the following estimate:

$$|a_nb_n - ab| = |a_nb_n - a_nb + a_nb - ab| \leq |a_n||b_n - b| + |b||a_n - a|. \tag{2.1}$$

Let $\varepsilon > 0$. Since $\{a_n\}$ converges to a, we may choose $N_1 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\varepsilon}{2(|b|+1)} \text{ for all } n \geq N_1.$$

Similarly, since $\{b_n\}$ converges to b, we may choose $N_2 \in \mathbb{N}$ such that

$$|b_n - b| < \frac{\varepsilon}{2M} \text{ for all } n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then, for $n \geq N$, it follows from (2.1) that

$$|a_nb_n - ab| < M\frac{\varepsilon}{2M} + |b|\frac{\varepsilon}{2(|b|+1)} < \varepsilon \text{ for all } n \geq N.$$

$$\frac{|b|\varepsilon}{2(|b|+1)} < \frac{|b|\varepsilon}{2|b|} = \frac{\varepsilon}{2}$$

Therefore, $\lim_{n\to\infty} a_nb_n = ab$. This proves (c).

(d) Let us first show that

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}.$$

Since $\{b_n\}$ converges to b, there is $N_1 \in \mathbb{N}$ such that

$$|b_n - b| < \frac{|b|}{2} \text{ for } n \geq N_1.$$

It follows (using a triangle inequality) that, for such n, $-\frac{|b|}{2} < |b_n| - |b| < \frac{|b|}{2}$ and, hence, $\frac{|b|}{2} < |b_n|. < \frac{3|b|}{2}$
For each $n \geq N_1$, we have the following estimate

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b_n||b|} \leq \frac{2|b_n - b|}{b^2}. \qquad \left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b_n b}\right| = \frac{|b - b_n|}{|b_n||b|} \tag{2.2}$$

$|b|^2 = b^2$

Now let $\varepsilon > 0$. Since $\lim_{n\to\infty} b_n = b$, there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - b| < \frac{b^2\varepsilon}{2} \text{ for all } n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. By (2.2), one has

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| \leq \frac{2|b_n - b|}{b^2} < \varepsilon \text{ for all } n \geq N.$$

It follows that $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$.

Finally, we can apply part (c) and have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \frac{1}{b_n} = \frac{a}{b}. \qquad a \cdot \frac{1}{b} = \frac{a}{b}.$$

The proof is now complete. □

$\frac{1}{n^\alpha}$, $\alpha = 2$

Recall $\lim_{n\to\infty}\left(\frac{1}{n^2}\right) = 0$ from previous section

$\lim_{n\to\infty}\left(\frac{5}{n^2}\right) = \lim_{n\to\infty}\left(5 \cdot \frac{1}{n^2}\right)$ $(k=5)$ by part (b.)

$= 5 \lim_{n\to\infty}\left(\frac{1}{n^2}\right)$

$= 5(0).$

Example 2.2.1 Consider the sequence $\{a_n\}$ given by

$$a_n = \frac{3n^2 - 2n + 5}{1 - 4n + 7n^2}. \tag{2.3}$$

Dividing numerator and denominator by n^2, we can write

$$a_n = \frac{3 - 2/n + 5/n^2}{1/n^2 - 4/n + 7} \tag{2.4}$$

Therefore, by the limit theorems above,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3 - 2/n + 5/n^2}{1/n^2 - 4/n + 7} = \frac{\lim_{n\to\infty} 3 - \lim_{n\to\infty} 2/n + \lim_{n\to\infty} 5/n^2}{\lim_{n\to\infty} 1/n^2 - \lim_{n\to\infty} 4/n + \lim_{n\to\infty} 7} = \frac{3}{7}. \tag{2.5}$$

Example 2.2.2 Let $a_n = \sqrt[n]{b}$, where $b > 0$. Consider the case where $b > 1$. In this case, $a_n > 1$ for every n. By the binomial theorem,

$$(a_n - 1 + 1)^n = \underbrace{(a_n - 1)^n + n(a_n-1)^{n-1} + \cdots + n(a_n - 1) + 1}_{\text{all positive}}$$

$$b = a_n^n = (a_n - 1 + 1)^n \geq 1 + n(a_n - 1).$$

This implies

$$0 < a_n - 1 \leq \frac{b-1}{n}.$$

$b - 1 \geq n(a_n - 1)$

$\frac{b-1}{n} \geq a_n - 1 > 0$ since $a_n > 1$

For each $\varepsilon > 0$, choose $N > \frac{b-1}{\varepsilon}$. It follows that for $n \geq N$,

$$|a_n - 1| = a_n - 1 < \frac{b-1}{n} \leq \frac{b-1}{N} < \varepsilon.$$

$a_n = \sqrt[n]{b}$, $b > 1$

$\lim\limits_{n \to \infty} \sqrt[n]{b} = 1$

Thus, $\lim_{n\to\infty} a_n = 1$.

In the case where $b = 1$, it is obvious that $a_n = 1$ for all n and, hence, $\lim_{n\to\infty} a_n = 1$.

If $0 < b < 1$, let $c = \frac{1}{b}$ and define

$$x_n = \sqrt[n]{c} = \frac{1}{a_n}.$$

Since $c > 1$, it has been shown that $\lim_{n\to\infty} x_n = 1$. This implies

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{x_n} = 1.$$

Exercises

2.2.1 Find the following limits:

(a) $\lim\limits_{n\to\infty} \dfrac{3n^2 - 6n + 7}{4n^2 - 3}$,

(b) $\lim\limits_{n\to\infty} \dfrac{1 + 3n - n^3}{3n^3 - 2n^2 + 1}$.

2.2.2 Find the following limits:

(a) $\lim\limits_{n\to\infty} \dfrac{\sqrt{3n} + 1}{\sqrt{n} + \sqrt{3}}$,

(b) $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{2n+1}{n}}$.

2.2.3 ▶ Find the following limits if they exist:

(a) $\lim_{n\to\infty}(\sqrt{n^2+n} - n)$.

(b) $\lim_{n\to\infty}(\sqrt[3]{n^3+3n^2} - n)$.

(c) $\lim_{n\to\infty}(\sqrt[3]{n^3+3n^2} - \sqrt{n^2+n})$.

(d) $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n})$.

(e) $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n})/n$.

2.2.4 Find the following limits.

(a) For $|r| < 1$ and $b \in \mathbb{R}$, $\lim_{n\to\infty}(b + br + br^2 + \cdots + br^n)$.

(b) $\lim_{n\to\infty} \left(\dfrac{2}{10} + \dfrac{2}{10^2} + \cdots + \dfrac{2}{10^n}\right)$.

2.2.5 Prove or disprove the following statements: