PHYSICS OF MOTION: KINEMATICS IN $\mathbb{R}^3$ VIA VECTORS

Def^{n}/

$\vec{r}$ = position

$\vec{v} = \frac{d\vec{r}}{dt}$ = velocity

$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt}$ = acceleration

$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$ = displacement

$\dot{s} = \frac{ds}{dt} = \|\vec{v}\|$ = speed

$s_{12} = \int_{t_1}^{t_2} \|\vec{v}(t)\|\,dt$ = distance travelled aka arclength

Relation with TNB discussion

$$\vec{v} = \dot{s}\,T \quad \left(\text{since } v = \frac{ds}{dt} \text{ and } T = \frac{1}{v}\vec{v} \;\therefore\; \vec{v} = v\,T = \dot{s}\,T\right)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{s}\,T) = \ddot{s}\,T + \dot{s}\frac{dT}{dt} = \ddot{s}\,T + \dot{s}(\kappa v N)$$

$$\boxed{\frac{dT}{dt} = \kappa v N}$$

Frenet Serret Eq^{n}

$$\therefore \quad \vec{a} = \ddot{s}\,T + \kappa(\dot{s})^2 N$$

$a_T = \vec{a}\cdot T = \ddot{s}$ = tangential acceleration

$a_N = \vec{a}\cdot N = \kappa v^2 = \kappa(\dot{s})^2$ = normal acceleration (centripetal)

$\left(a_c = \frac{v^2}{R} \text{ in Physics}\right)$

$$\underbrace{\|\vec{a}\|^2 = a_T^2 + a_N^2}$$

can be useful to find a_T or a_N when the other is already known.

Example: given $\vec{V}(t) = \langle t^2, \cos(t), \sin(6t) \rangle$ and the position at $t = 1$ is $(4, 0, 0)$ find the position and acceleration as fnct. of t. Also find $\boxed{a_T}$ & $\boxed{a_N}$

Acceleration is easy, $\boxed{\vec{a} = \frac{d\vec{V}}{dt} = \langle 2t, -\sin t, 6\cos(6t) \rangle.}$

Position follows from integration; $\int_1^t \frac{d\vec{r}}{d\tau}\, d\tau = \int_1^t \vec{V}(\tau)\, d\tau$

thus $\vec{r}(t) - \vec{r}(1) = \int_1^t \langle \tau^2, \cos\tau, \sin 6\tau \rangle\, d\tau$

$$\Rightarrow \vec{r}(t) = \vec{r}(1) + \left\langle \frac{\tau^3}{3}, \sin\tau, \frac{-1}{6}\cos 6\tau \right\rangle \Big|_1^t$$

$$\therefore \boxed{\vec{r}(t) = (4, 0, 0) + \left\langle \tfrac{1}{3}(t^3 - 1), \sin(t) - \sin(1), \tfrac{-1}{6}[\cos(6t) - \cos(6)] \right\rangle}$$

To find tangential acceleration, notice $\dot{s} = \|\vec{V}\| = \sqrt{t^4 + \cos^2 t + \sin^2(6t)}$

thus $a_T = \ddot{s} = \frac{d}{dt}\left(\sqrt{t^4 + \cos^2 t + \sin^2(6t)}\right) = \boxed{\dfrac{4t^3 - 2\sin t\cos t + 12\cos(6t)\sin(6t)}{2\sqrt{t^4 + \cos^2 t + \sin^2 6t}}}$

Then $a_N = \sqrt{a^2 - a_T^2} = \underbrace{\sqrt{4t^2 + \sin^2 t + 36\cos^2(6t) - \left[\dfrac{4t^3 - 2\sin t\cos t + 12\cos 6t\sin 6t}{2\sqrt{t^4 + \cos^2 t + \sin^2 6t}}\right]^2}}_{a_N = \kappa v^2}$

$a^2 = a_T^2 + a_N^2$