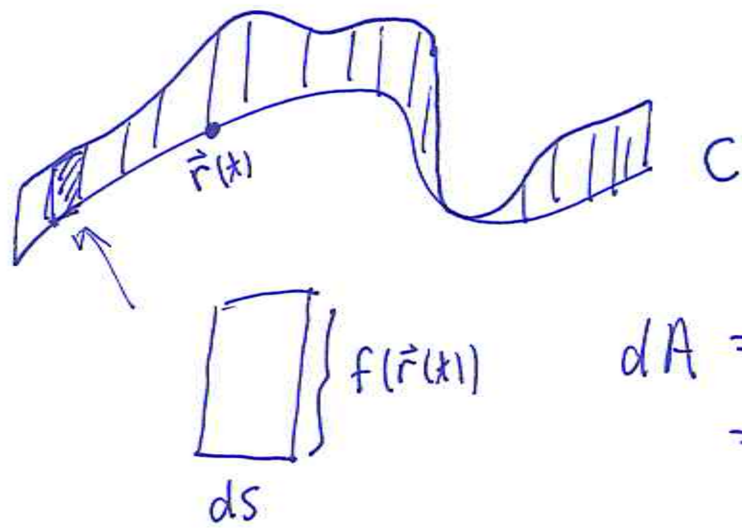
INTEGRATION OF SCALAR FUNCTION ALONG CURVE

$f(\vec{r}(t))$ = height of curtain at $\vec{r}(t)$

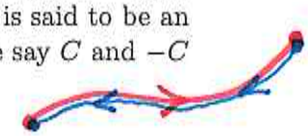


$$dA = f(\vec{r}(t))\,ds$$
$$= \underbrace{f(\vec{r}(t))\frac{ds}{dt}\,dt}_{\text{integrate over } t}$$

Remark: There is another method of proving Kepler's Laws that begins with the two-body Lagrangian for a central potential (well force really but $\vec{F} = f(r)\hat{r} \Rightarrow U = U(r) \dots$). In that derivation one need not assume the sun is at the origin. Instead you consider the center of mass to be at the origin and work out how the reduced mass μ orbits. Anyway its very beautiful, take Mechanics at the Junior/Senior level to see the more general derivation. Also they will actually find $r(t)$ explicitly as opposed to the indirect arguments we have offered (or rather stolen from Colley ☺.)

2.4 integration of scalar function along a curve

In this section we learn how to sum a quantity along some curve. Let's begin by reviewing some terminology. A **path** in $\mathbb{R}^3$ is a continuous function $\vec{\gamma}$ with connected domain I such that $\vec{\gamma} : I \subset \mathbb{R} \to \mathbb{R}^3$. If $\partial I = \{a, b\}$ then we say that $\vec{\gamma}(a)$ and $\vec{\gamma}(b)$ are the endpoints of the path $\vec{\gamma}$. When $\vec{\gamma}$ has continuous derivatives of all orders we say it is a smooth path (of class C^∞), if it has at least one continuous derivative we say it is a differentiable path(of class C^1). When $I = [a, b]$ then the path is said to go from $\vec{\gamma}(a) = P$ to $\vec{\gamma}(b) = Q$ and the image $C = \vec{\gamma}([a, b])$ is said to be an **oriented curve** C from P to Q. The same curve from Q to P is denoted $-C$. We say C and $-C$ have opposite orientations.



Hopefully most of this is already familar from our earlier work on parametrizations. I give another example just in case.

Example 2.4.1. *The line-segment L from $(1, 2, 3)$ to $(5, 5, 5)$ has parametric equations $x = 1 + 4t, y = 2 + 3t, z = 3 + 2t$ for $0 \leq t \leq 1$. In other words, the path $\vec{\gamma}(t) = \langle 1 + 4t, 2 + 3t, 3 + 2t \rangle$ covers the line-segment L. In contrast $-L$ goes from $(5, 5, 5)$ to $(1, 2, 3)$ and we can parametrize it by $x = 5-4u, y = 5-3u, z = 5-2u$ or in terms of a vector-formula $\vec{\gamma}_{reverse}(u) = \langle 5-4u, 5-3u, 5-2u \rangle$. How are these related? Observe:*

$$\vec{\gamma}_{reverse}(0) = \vec{\gamma}(1) \qquad \& \qquad \vec{\gamma}_{reverse}(1) = \vec{\gamma}(0)$$

Generally, $\vec{\gamma}_{reverse}(t) = \vec{\gamma}(1 - t)$.

We can generalize this construction to other curves. If we are given C from P to Q parametrized by $\vec{\gamma} : [a, b] \to \mathbb{R}^3$ then we can parametrize $-C$ by $\vec{\gamma}_{reverse} : [a, b] \to \mathbb{R}^3$ defined by $\vec{\gamma}_{reverse}(t) = \vec{\gamma}(a + b - t)$. Clearly we have $\vec{\gamma}_{reverse}(a) = \vec{\gamma}(b) = Q$ whereas $\vec{\gamma}_{reverse}(b) = \vec{\gamma}(a) = Q$. Perhaps it is interesting to compare these paths at a common point,

$$\gamma(t) = \vec{\gamma}_{reverse}(a + b - t)$$

The velocity vectors naturally point in opposite directions, (by the chain-rule)

$$\frac{d\vec{\gamma}}{dt}(t) = -\frac{d\vec{\gamma}_{reverse}}{dt}(a + b - t).$$

Example 2.4.2. *Suppose $\vec{\gamma}(t) = \langle \cos(t), \sin(t) \rangle$ for $\pi \leq t \leq 2\pi$ covers the oriented curve C. If we wish to parametrize $-C$ by $\vec{\beta}$ then we can use*

$\beta(\pi) = \vec{\gamma}(2\pi) = (1,0)$
$\beta(2\pi) = \vec{\gamma}(\pi) = (-1,0)$

$$\vec{\beta}(t) = \vec{\gamma}(3\pi - t) = \langle \cos(3\pi - t), \sin(3\pi - t) \rangle$$

Simplifying via trigonometry yields $\vec{\beta}(t) = \langle -\cos(t), -\sin(t) \rangle$ for $\pi \leq t \leq 2\pi$. You can easily verify that $\vec{\beta}$ covers the lower half of the unit-circle in a CW-fashion, it goes from (1.0) *to* $(-1, 0)$

What I have just described is a general method to reverse a path whilst keeping the same domain for the new path. Naturally, you might want to use a different domain after you change the parametrization of a given curve. Let's settle the general idea with a definition. This definition describes what we allow as a reasonable reparametrization of a curve.

$u(t) = 3\pi - t$

Definition 2.4.3. $\vec{\gamma}_1(t) = \vec{\gamma}_2(u(t))$

$\frac{du}{dt} = -1$

Let $\vec{\gamma}_1 : [a_1, b_1] \to \mathbb{R}^3$ be a path. We say another path $\vec{\gamma}_2 : [a_2, b_2] \to \mathbb{R}^3$ is a **reparametrization** of $\vec{\gamma}_1$ if there exists a bijective (one-one and onto), continuous function $u : [a_1, b_1] \to [a_2, b_2]$ with continuous inverse $u^{-1} : [a_2, b_2] \to [a_1, b_1]$ such that $\vec{\gamma}_1(t) = \vec{\gamma}_2(u(t))$ for all $t \in [a_1, b_1]$. If the given curve is smooth or k-times differentiable then we also insist that the transition function u and its inverse be likewise smooth or k-times differentiable.

In short, we want the allowed reparametrizations to capture the same curve without adding any artificial stops, starts or multiple coverings. If the original path wound around a circle 10 times then we insist that the allowed reparametrizations also wind 10 times around the circle. Finally, let's compare the a path and its reparametrization's velocity vectors, by the chain rule we find:

$$\vec{\gamma}_1(t) = \vec{\gamma}_2(u(t)) \qquad \Rightarrow \qquad \frac{d\vec{\gamma}_1}{dt}(t) = \frac{du}{dt}\frac{d\vec{\gamma}_2}{dt}(u(t)).$$

This calculation is important in the section that follows. Observe that:

1. if $du/dt > 0$ then the paths progress in the same direction and are **consistently oriented**
2. if $du/dt < 0$ then the paths go in opposite directions and are **oppositely oriented**

Reparametrizations with $du/dt > 0$ are said to be **orientation preserving**.

2.4.1 line-integral of scalar function

These are also commonly called the **integral with respect to arclength**. In lecture we framed the need for this definition by posing the question of finding the area of a curved fence with height $f(x, y)$. It stood to reason that the infinitesimal area dA of the curved fence over the arclength ds would simply be $dA = f(x, y)ds$. Then integration is used to sum all the little areas up. Moreover, the natural calculation to accomplish this is clearly as given below:

Definition 2.4.4.

Let $\vec{\gamma} : [a, b] \to C \subset \mathbb{R}^n$ be a differentiable path and suppose that $C \subset dom(f)$ for a continuous function $f : dom(f) \to \mathbb{R}$ then the **scalar line integral of f along C** is

$$\int_C f\, ds \equiv \int_a^b f(\vec{\gamma}(t))\, ||\vec{\gamma}\,'(t)||\, dt. \quad \leftarrow f(\ )\frac{ds}{dt}dt$$

We should check to make sure there is no dependence on the choice of parametrization above. If there was then this would not be a reasonable definition. Suppose $\vec{\gamma}_1(t) = \vec{\gamma}_2(u(t))$ for $a_1 \leq t \leq b_1$ where $u : [a_1, b_1] \to [a_2, b_2]$ is differentiable and strictly monotonic. Note

$$\begin{aligned}\int_{a_1}^{b_1} f(\vec{\gamma}_1(t)) \left|\left|\frac{d\vec{\gamma}_1}{dt}\right|\right| dt &= \int_{a_1}^{b_1} f(\vec{\gamma}_2(u(t))) \left|\left|\frac{du}{dt}\frac{d\vec{\gamma}_2}{dt}(u(t))\right|\right| dt \\ &= \int_{a_1}^{b_1} f(\vec{\gamma}_2(u(t))) \left|\left|\frac{d\vec{\gamma}_2}{dt}(u(t))\right|\right| \cdot \left|\frac{du}{dt}\right| dt\end{aligned}$$

If u is orientation preserving then $du/dt > 0$ hence $u(a_1) = a_2$ and $u(b_1) = b_2$ and thus

$$\begin{aligned}\int_{a_1}^{b_1} f(\vec{\gamma}_1(t)) \left|\left|\frac{d\vec{\gamma}_1}{dt}\right|\right| dt &= \int_{a_1}^{b_1} f(\vec{\gamma}_2(u(t))) \left|\left|\frac{d\vec{\gamma}_2}{dt}(u(t))\right|\right| \frac{du}{dt}\, dt \\ &= \int_{a_2}^{b_2} f(\vec{\gamma}_2(u) \left|\left|\frac{d\vec{\gamma}_2}{du}\right|\right| du.\end{aligned}$$

On the other hand, if $du/dt < 0$ then $|du/dt| = -du/dt$ and the bounds flip since $u(a_1) = b_2$ and $u(b_1) = a_2$

$$\begin{aligned}\int_{a_1}^{b_1} f(\vec{\gamma}_1(t)) \left|\left|\frac{d\vec{\gamma}_1}{dt}\right|\right| dt &= -\int_{a_1}^{b_1} f(\vec{\gamma}_2(u(t))) \left|\left|\frac{d\vec{\gamma}_2}{dt}(u(t))\right|\right| \frac{du}{dt}\, dt \\ &= -\int_{b_2}^{a_2} f(\vec{\gamma}_2(u) \left|\left|\frac{d\vec{\gamma}_2}{du}\right|\right| du. \\ &= \int_{a_2}^{b_2} f(\vec{\gamma}_2(u) \left|\left|\frac{d\vec{\gamma}_2}{du}\right|\right| du.\end{aligned}$$

Note, the definition requires me to flip the bounds before I judge if we have the same result. This is implicit in the statement in the definition that $dom(\vec{\gamma}) = [a, b]$ this forces $a < b$ and hence the integral in turn. Technical details aside we have derived the following important fact:

$$\boxed{\int_C f ds = \int_{-C} f ds} \qquad \int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r}$$

The scalar-line integral of function with no attachment to C is independent of the orientation of the curve. Given our original motivation for calculating the area of a curved fence this is not surprising.

One convenient notation calculation of the scalar-line integral is given by the dot-notation of Newton. Recall that $dx/dt = \dot{x}$ hence $\vec{\gamma} = \langle x, y, z \rangle$ has $\vec{\gamma}\,'(t) = \langle \dot{x}, \dot{y}, \dot{z} \rangle$. Thus, for a space curve,

$$\int_C f \; ds \equiv \int_a^b f(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt.$$

We can also calculate the scalar line integral of f along some curve which is made of finitely many differentiable segments, we simply calculate each segment's contribution and sum them together. Just like calculating the integral of a piecewise continuous function with a finite number of jump-discontinuities, you break it into pieces.

Furthermore, notice that if we calculate the scalar line integral of the constant function $f = 1$ then we will obtain the arclength of the curve. More generally the scalar line integral calculates the weighted sum of the values that the function f takes over the curve C. If we divide the result by the length of C then we would have the average of f over C.

Example 2.4.5. *Suppose the linear mass density of a helix $x = R\cos(t), y = R\sin(t), z = t$ is given by $dm/dz = z$. Calculate the total mass around the two twists of the helix given by $0 \leq t \leq 4\pi$.*

$\frac{dm}{ds} = z$

$dm = z ds$

$M_{TOTAL} = \int_C dm$

$$m_{total\ on\ C} = \int_C z ds = \int_0^{4\pi} z\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\; dt \tag{2.1}$$
$$= \int_0^{4\pi} t\sqrt{R^2 + 1}\; dt$$
$$= \frac{t^2\sqrt{R^2+1}}{2}\bigg|_0^{4\pi}$$
$$= \boxed{8\pi^2\sqrt{R^2+1}.}$$

In contrast to total mass we could find the arclength by simply adding up ds, the total length L of C is given by

$$L = \int_C ds = \int_0^{4\pi} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\; dt$$
$$= \int_0^{4\pi} \sqrt{R^2 + 1}\; dt$$
$$= \boxed{4\pi\sqrt{R^2+1}.}$$

Definition 2.4.6.

Let C be a curve with length L then the average of f over C is given by

$$f_{avg} = \frac{1}{L}\int_C f \; ds.$$

Example 2.4.7. *The average mass per unit length of the helix with $dm/dz = z$ as studied above is given by*

$$m_{avg} = \frac{1}{L}\int_C f \; ds = \frac{1}{4\pi\sqrt{R^2+1}} 8\pi^2\sqrt{R^2+1} = \boxed{2\pi.}$$

Since $z = t$ and $0 \leq t \leq 4\pi$ over C this result is hardly surprising.

Another important application of the scalar line integral is to find the center of mass of a wire. The idea here is nearly the same as we discussed for volumes, the difference is that the mass is distributed over a one-dimensional space so the integration is one-dimensional as opposed to two-dimensional to find the center of mass for a planar laminate or three-dimensional to find the center of mass for a volume.

Definition 2.4.8.

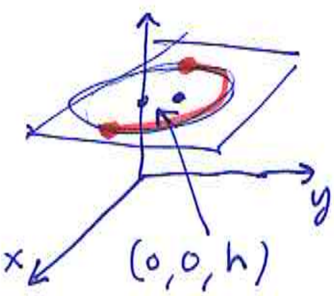
Let C be a curve with length L and suppose $dM/ds = \delta$ is the mass-density of C. The total mass of the curve found by $M = \int_c \delta ds$. The **centroid** or **center of mass** for C is found at $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{1}{M}\int_C x\delta \, ds, \qquad \bar{y} = \frac{1}{M}\int_C y\delta \, ds, \qquad \bar{z} = \frac{1}{M}\int_C z\delta \, ds.$$

Often the centroid is found off the curve.

Example 2.4.9. *Suppose $x = R\cos(t), y = R\sin(t), z = h$ for $0 \leq t \leq \pi$ for a curve with $\delta = 1$. Clearly $ds = Rdt$ and thus $M = \int_C \delta ds = \int_0^\pi Rdt = \pi R$. Consider,*

$$\bar{x} = \frac{1}{\pi R}\int_C xds = \frac{1}{\pi R}\int_0^\pi R^2\cos(t)dt = 0$$



whereas,

$$\bar{y} = \frac{1}{\pi R}\int_C yds = \frac{1}{\pi R}\int_0^\pi R^2\sin(t)dt = \frac{1}{\pi R}(-R^2\cos(t)\Big|_0^\pi = \frac{2R}{\pi}$$

The reader can easily verify that $\bar{z} = h$ hence the centroid is at $(0, \frac{2R}{\pi}, h)$.

Of course there are many other applications, but I believe these should suffice for our current purposes. We will eventually learn that $\int_C \vec{F} \cdot \vec{T} ds$ and $\int_C \vec{F} \cdot \vec{N} ds$ are also of interest, but we should cover other topics before returning to these. Incidentally, it is pretty obvious that we have the following properties for the scalar-line integral:

$$\int_C (f + cg)ds = \int_C fds + c\int_C gds \qquad \& \qquad \int_{C\cup\bar{C}} fds = \int_C fds + \int_{\bar{C}} fds$$

in addition if $f \leq g$ on C then $\int_C fds \leq \int_C gds$. I leave the proof to the reader.

Remark 2.4.10.

I have a few solved problems on integrals along a curve and centroids. They are attached to a later Chapter. See Problems 187, 188, 189.

2.5 Problems

Problem 46 Calculate the following:

(a.) $\frac{d}{dt}\langle t^2, e^t, \ln(t)\rangle$

(b.) $\frac{d}{dt}\langle \cosh(t^2), \sinh(\ln(t))\rangle$

(c.) $\int \langle 1, t, \sin(t)\rangle dt$