LECTURE 11 : MONOTONE SEQUENCES

Defn/ A sequence $\{a_n\}$ is *increasing* if $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$.
A sequence $\{a_n\}$ is *decreasing* if $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$.
If $\{a_n\}$ is increasing or decreasing then $\{a_n\}$ is a *monotone sequence.*

If $a_n < a_{n+1}$ or $a_n > a_{n+1}$ $\forall n \in \mathbb{N}$ then such sequences are respectively *strictly increasing* or *strictly decreasing*. Notice $a_n \leq a_m$ whenever $n < m$ for an increasing sequence.

Thm (Bounded Monotonic Sequence Theorem)
Let $\{a_n\}$ be sequence in $\mathbb{R}$ then,
(a.) If $\{a_n\}$ is inc. and bounded above, then $\{a_n\}$ is convergent : $\lim_{n\to\infty}(a_n) = \sup\{a_n \mid n \in \mathbb{N}\}$
(b.) If $\{a_n\}$ is dec. and bounded below, then $\{a_n\}$ is convergent. : $\lim_{n\to\infty}(a_n) = \inf\{a_n \mid n \in \mathbb{N}\}$

Proof: Let $\{a_n\}$ be an inc. sequence which is bounded above. Construct

$$A = \{a_n \mid n \in \mathbb{N}\}$$

then $\sup(A) \in \mathbb{R}$ since $A \neq \emptyset$ and $A \subseteq \mathbb{R}$ is bounded above.
Let $l = \sup(A)$ and suppose $\varepsilon > 0$ then $\exists N \in \mathbb{N}$ such that $l - \varepsilon < a_N \leq l$
Moreover, as $\{a_n\}$ is inc, $l - \varepsilon < a_N \leq a_n$ $\forall n \geq N$. Still, $a_n \in A$ so $a_n \leq l < l + \varepsilon$.
Thus $l - \varepsilon < a_n < l + \varepsilon$ $\forall n \geq N$. That is $|a_n - l| < \varepsilon$ $\forall n \geq N$ $\therefore a_n \to l$ as $n \to \infty$. #

To prove (b.) if a_n is dec. and bounded below then $-a_n$ is increasing & bounded above
hence $-a_n \longrightarrow L$ for $L = \sup\{-a_n\}$. Recall $\sup\{-a_n \mid n \in \mathbb{N}\} = -\inf\{a_n \mid n \in \mathbb{N}\}$
hence $-a_n \longrightarrow -\inf\{a_n \mid n \in \mathbb{N}\}$ and so $a_n \longrightarrow \inf\{a_n \mid n \in \mathbb{N}\}$. (I'm using some hwk's from a previous section)

Example 2.3.1: Let $r \in \mathbb{R}$ with $|r|<1$ then $\lim_{n\to\infty}(r^n) = 0$

If $r=0$ then $a_n = r^n = 0 \longrightarrow 0$ is clear.

If $0<|r|<1$ then $a_n = r^n$ has $|a_n| = |r|^n$. Let $b_n = |a_n|$

Observe $b_{n+1} = |r|^{n+1} = |r|\,|r|^n = |r|\,b_n \therefore b_{n+1} < b_n \ \forall n \in \mathbb{N} \therefore \{b_n\}$ dec.

Moreover, $0 < |r|^n = b_n$ so $\{b_n\}$ is dec. & bounded below $\therefore \lim_{n\to\infty} |r|^n = l$.

or $\lim_{n\to\infty}(b_n) = l$. Notice $b_{n+1} = |r|\,b_n \longrightarrow l = |r|\,l$ as $n \to \infty$ *

hence $l(1-|r|) = 0$ where $|r| \neq 1$ thus $l = 0$ and we've shown $\lim_{n\to\infty}|a_n| = 0$.

But, $\underbrace{|a_n| \to 0 \Rightarrow a_n = r^n}_{\text{hwk exercise 2.1.3}} \longrightarrow 0$ so the result follows.

Example 2.3.2: Define $a_1 = 2$ and $a_{n+1} = \dfrac{a_n+5}{3}$ for $n \geq 1$. We argue $a_n \to \frac{5}{2}$

- Induction on n shows a_n is inc. Notice $a_2 = \frac{a_1+5}{3} = \frac{2+5}{3} = \frac{7}{3} > 2 = a_1$ thus $n=1$ for $a_{n+1} > a_n$ is true. Suppose $a_{n+1} > a_n$ for some $n \in \mathbb{N}$. Consider
$$a_{n+2} = \frac{a_{n+1}+5}{3} > \frac{a_n+5}{3} = a_{n+1} \quad \therefore \quad a_{n+1} < a_{n+2}$$ hence claim true for $n+1$.
Therefore, by PMI, $a_{n+1} > a_n \ \forall n \in \mathbb{N}$.
- Induction on n shows a_n is bounded above by 3. Clearly $a_1 = 2 < 3$. Suppose $a_n \leq 3$ for some $n \in \mathbb{N}$. Then $a_{n+1} = \frac{a_n+5}{3} \leq \frac{3+5}{3} = \frac{8}{3} < \frac{9}{3} = 3$. thus $a_n \leq 3$ by PMI.
- $\{a_n\}$ is inc. and bounded above $\therefore a_n \longrightarrow l$ by Bounded Monotonic Sequence Thm.
- $a_{n+1} = \dfrac{a_n+5}{3} \longrightarrow l = \dfrac{l+5}{3} \therefore 3l = l+5 \Rightarrow \underline{l = \frac{5}{2}}$. * same trick

Example 2.3.3: Let $a_n = \left(1+\frac{1}{n}\right)^n$ for $n \in \mathbb{N}$

We proved the Binomial Th^{m} in a previous lecture. Let's use it here,

$$a_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}\left(\frac{1}{n}\right)^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

$$= 1 + 1 + \frac{1}{2!}\left(\frac{n(n-1)}{n^2}\right) + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots + \frac{n(n-1)\cdots(n-(n-1))}{n!}\frac{1}{n^n}$$

$$= \underline{1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)} \quad *$$

Likewise,

$$a_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{(n+1)-1}{n+1}\right)$$

Notice $n+1 > n$ and so $1-\frac{1}{n+1} > 1-\frac{1}{n}$ and $1-\frac{2}{n} > 1-\frac{2}{n+1}$ etc. Thus, $\quad a_{n+1} > a_n$.
(a_{n+1} also has one additional term with the $\frac{1}{(n+1)!}$ coefficient, that only helps show $a_{n+1} > a_n$)

Consider, from * it is clear that,

$$a_n \le 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} < 2 + \underbrace{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)\cdot n}}_{\sum_{k=1}^{n-1}\frac{1}{k(k+1)}}$$

(made denominators smaller hence the fractions larger)

$$\sum_{k=1}^{n-1}\frac{1}{k(k+1)} = \sum_{k=1}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n-1+1}\right)$$

$$= 1-\frac{1}{n} \quad \text{(telescopes)}$$

Thus,

$$a_n < 3 - \frac{1}{n} < 3$$

Thus $a_n = \left(1+\frac{1}{n}\right)^n$ is a bounded monotonic sequence and thus $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \in \mathbb{R}$

Btw, $\quad e = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots = \sum_{k=0}^{\infty}\frac{1}{k!}$

Th$^{\underline{m}}$ 2.3.3: NESTED INTERVALS THEOREM

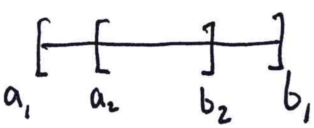
Let $\{I_n\}$ be a sequence of nonempty, closed & bounded intervals satisfying the <u>nesting</u> condition $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$. Then the following hold:

(a.) $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$

(b.) If the lengths of the intervals I_n converge to zero then $\bigcap_{n=1}^{\infty} I_n = \{x_o\}$ for some point $x_o \in \mathbb{R}$.

<u>Proof:</u> (a.) Let $I_n = [a_n, b_n]$ where $I_{n+1} \subseteq I_n \ \forall n \in \mathbb{N}$. Then $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \Rightarrow a_{n+1} \geq a_n$ and $b_{n+1} \leq b_n \ \forall n \in \mathbb{N}$.

Then observe $\underbrace{a_1 \leq a_2 \leq \cdots}_{\text{increasing}}$ whereas $\underbrace{b_1 \geq b_2 \geq \cdots}_{\text{decreasing}}$



$a_n \leq b_1$ bounded above by b_1

$b_n \geq a_1$ bounded below by a_1

Thus by monotonic sequence Th$^{\underline{m}}$ $a_n \longrightarrow a_* \in \mathbb{R}$ and $b_n \longrightarrow b_* \in \mathbb{R}$

We argue $[a_*, b_*] = \bigcap_{n=1}^{\infty} I_n$. Recall $a_* = \sup\{a_n \mid n \in \mathbb{N}\}$ from proof of Bounded M.S. Th$^{\underline{m}}$.

thus $a_* \geq a_n \ \forall n \in \mathbb{N}$. Likewise $b_* = \inf\{b_n \mid n \in \mathbb{N}\} \therefore b_* \leq b_n \ \forall n \in \mathbb{N}$.

Thus $[a_n, b_n] \supseteq [a_*, b_*] \ \forall n \in \mathbb{N}$ so $[a_*, b_*] \subseteq \bigcap_{n=1}^{\infty} I_n$.

Likewise, if $x \in \bigcap_{n=1}^{\infty} I_n$ then $x \in [a_n, b_n] \ \forall n \in \mathbb{N}$ hence $a_n \leq x \leq b_n$

thus by Squeeze Th$^{\underline{m}}$ $a_* \leq \lim_{n\to\infty}(x) \leq b_* \Rightarrow a_* \leq x \leq b_* \therefore x \in [a_*, b_*]$

Consequently, $\bigcap_{n=1}^{\infty} I_n \subseteq [a_*, b_*]$ and it follows $\bigcap_{n=1}^{\infty} I_n = [a_*, b_*]$.

(b.) If length of $[a_n, b_n]$ goes to 0 then $b_n - a_n \longrightarrow 0$ thus $b_n - a_n \longrightarrow b_* - a_* = 0$

$\therefore a_* = b_*$ //

Divergent Sequences which tend to $\pm\infty$

Defn/ A sequence $\{a_n\}$ is said to diverge to ∞ if for every $M \in \mathbb{R}$ $\exists N \in \mathbb{N}$ such that $a_n > M$ $\forall n \geq N$. Likewise $\{a_n\}$ diverges to $-\infty$ if for every $M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ s.t. $a_n < M$ $\forall n \geq N$. We write $\lim_{n\to\infty}(a_n) = \infty$ when $\{a_n\}$ diverges to ∞, and $\lim_{n\to\infty}(a_n) = -\infty$ when $\{a_n\}$ diverges to $-\infty$.

<u>Remark:</u> If $\lim_{n\to\infty} a_n \notin \mathbb{R}$ then $\{a_n\}$ does not converge, aka $\{a_n\}$ is divergent. If $\lim_{n\to\infty} a_n = \pm\infty$ then $\{a_n\}$ is divergent, but, there is more structure here. I suggest we think about $a_n \longrightarrow \pm\infty$ as a special type of divergence.

<u>Example:</u> Let $a_n = n^2$. Suppose $M \in \mathbb{R}$ if $M \leq 0$ then $a_n = n^2 > M$ $\forall n \in \mathbb{N}$. Suppose $M > 0$ then let $\lceil x \rceil$ = the next greater integer to x. For example $\lceil 3.01 \rceil = 4$. Let $N = \lceil \sqrt{M} \rceil$ then $N^2 = (\lceil \sqrt{M} \rceil)^2 \geq (\sqrt{M})^2 = M$. If $n \geq N$ then $a_n = n^2 \geq N^2 \geq M$. <u>Oops, I've allowed = in my construction</u>. Let's fix it. Let $\bar{N} = \lceil \sqrt{M} \rceil + 1$ then $\bar{N}^2 = (\lceil \sqrt{M} \rceil + 1)^2 \geq (\sqrt{M})^2 + 2\sqrt{M} + 1 > M + 1$ so if $n \geq \bar{N}$ then $a_n = n^2 \geq \bar{N}^2 > M + 1 > M$ thus $\lim_{n\to\infty}(n^2) = \infty$ //

Th^{m} (2.3.5) If $\{a_n\}$ is increasing and not bounded above then $\lim_{n\to\infty}(a_n) = \infty$. Likewise if $\{a_n\}$ is dec. and not bounded below then $\lim_{n\to\infty}(a_n) = -\infty$.

<u>Proof</u>: Let M be a fixed real #. Since $\{a_n\}$ is not bounded above $\exists N \in \mathbb{N}$ such that $a_n > M$. Thus $a_n \geq a_N > M$ $\forall$ $n \geq N$ as $\{a_n\}$ inc. thus $\lim_{n\to\infty}(a_n) = \infty$. (this proof suffers the same flaw as my Example on 5) (if you use the text's proof, I modified it slightly)

I leave the last half to the reader ☺. //

Likewise, I'll let you read the text for the proof of the following:

Th^{m} (2.3.6)

Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences in $\mathbb{R}$ and k be constant

Suppose $a_n \to \infty$, $b_n \to \infty$ and $c_n \to -\infty$. Then,

(a.) $a_n + b_n \to \infty$

(b.) $a_n b_n \to \infty$

(c.) $a_n c_n \to -\infty$

(d.) $k a_n \to \pm\infty$ depending on sign of k $(k > 0, \infty)(k < 0, -\infty)$

(e.) $\frac{1}{a_n} \to 0$ (provided $a_n \neq 0$ $\forall n \in \mathbb{N}$)

Th^{m} (2.3.7) COMPARISON

If $a_n \leq b_n$ $\forall n \in \mathbb{N}$ then
(a.) If $a_n \to \infty$ then $b_n \to \infty$
(b.) If $b_n \to -\infty$ then $a_n \to -\infty$