LECTURE 12: BOLZANO-WEIERSTRASS THEOREM

Th^m 2.4.1 (BOLZANO-WEIERSTRASS)
Every bounded sequence of real numbers has a convergent subsequence.

Proof: Suppose $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$. Define $A = \{a_n \mid n \in \mathbb{N}\}$.

- If A is finite then $\exists x \in A$ such that $a_n = x$ for infinitely many n. Let n_1 be the smallest $n \in \mathbb{N}$ for which $a_n = x$ (use WOP) then n_2 is selected s.t. $a_{n_2} = x$ where $n_2 > n_1$ (once again, can consider set of all $n \in \mathbb{N}$ for which $a_n = x$ call this S_x so n_1 is smallest element of S_x and n_2 is smallest element of $S_x - \{n_1\}$, we can continue in this fashion w/o end since S_x is infinite)
- If A is infinite then we note $c \leq a_n \leq d$ $\forall n \in \mathbb{N}$ as $\{a_n\}$ is assumed bounded. We now define a sequence of nested intervals given by targeted bisection. Begin with $I_1 = [c,d]$ then since A infinite at least one of $A \cap [\frac{c+d}{2}, d]$ and $A \cap [c, \frac{c+d}{2}]$ is infinite and we choose an infinite subset to define I_2. Then I_3 is selected in the same fashion, continuing I_n has infinitely many points in A. Moreover, $I_n \subseteq I_{n-1}$ and the length of $I_n \longrightarrow 0$ by construction. Thus $\bigcap_{n=1}^{\infty} I_n = \{x_0\}$ by Nested Intervals Th^m of §2.3.

2.4 THE BOLZANO-WEIERSTRASS THEOREM

The Bolzano-Weierstrass Theorem is at the foundation of many results in analysis. It is, in fact, equivalent to the completeness axiom of the real numbers.

Theorem 2.4.1 — Bolzano-Weierstrass. Every bounded sequence $\{a_n\}$ of real numbers has a convergent subsequence.

Proof: Suppose $\{a_n\}$ is a bounded sequence. Define $A = \{a_n : n \in \mathbb{N}\}$ (the set of values of the sequence $\{a_n\}$). If A is finite, then at least one of the elements of A, say x, must be equal to a_n for infinitely many choices of n. More precisely, $B_x = \{n \in \mathbb{N} : a_n = x\}$ is infinite. We can then define a convergent subsequence as follows. Pick n_1 such that $a_{n_1} = x$. Now, since B_x is infinite, we can choose $n_2 > n_1$ such that $a_{n_2} = x$. Continuing in this way, we can define a subsequence $\{a_{n_k}\}$ which is constant, equal to x and, thus, converges to x.

Suppose now that A is infinite. First observe there exist $c, d \in \mathbb{R}$ such that $c \le a_n \le d$ for all $n \in \mathbb{N}$, that is, $A \subset [c, d]$.

We define a sequence of nonempty nested closed bounded intervals as follows. Set $I_1 = [c, d]$. Next consider the two subintervals $[c, \frac{c+d}{2}]$ and $[\frac{c+d}{2}, d]$. Since A is infinite, at least one of $A \cap [c, \frac{c+d}{2}]$ or $A \cap [\frac{c+d}{2}, d]$ is infinite. Let $I_2 = [c, \frac{c+d}{2}]$ if $A \cap [c, \frac{c+d}{2}]$ is infinite and $I_2 = [\frac{c+d}{2}, d]$ otherwise. Continuing in this way, we construct a nested sequence of nonempty closed bounded intervals $\{I_n\}$ such that $I_n \cap A$ is infinite and the length of I_n tends to 0 as $n \to \infty$.

We now construct the desired subsequence of $\{a_n\}$ as follows. Let $n_1 = 1$. Choose $n_2 > n_1$ such that $a_{n_2} \in I_2$. This is possible since $I_2 \cap A$ is infinite. Next choose $n_3 > n_2$ such that $a_{n_3} \in I_3$. In this way, we obtain a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \in I_k$ for all $k \in \mathbb{N}$.

Set $I_n = [c_n, d_n]$. Then $\lim_{n\to\infty}(d_n - c_n) = 0$. We also know from the proof of the Monotone Convergence Theorem (Theorem 2.3.1), that $\{c_n\}$ converges. Say $\ell = \lim_{n\to\infty} c_n$. Thus, $\lim_{n\to\infty} d_n = \lim_{n\to\infty}[(d_n - c_n) + c_n] = \ell$ as well. Since $c_k \le a_{n_k} \le d_k$ for all $k \in \mathbb{N}$, it follows from Theorem 2.1.5 that $\lim_{k\to\infty} a_{n_k} = \ell$. This completes the proof. □

c_k [a_{n_k}] d_k $\quad I_k = [c_k, d_k]$

Definition 2.4.1 (Cauchy sequence). A sequence $\{a_n\}$ of real numbers is called a *Cauchy sequence* if for any $\varepsilon > 0$, there exists a positive integer N such that for any $m, n \ge N$, one has

$$|a_m - a_n| < \varepsilon.$$

Theorem 2.4.2 A convergent sequence is a Cauchy sequence.

Proof: Let $\{a_n\}$ be a convergent sequence and let

$$\lim_{n\to\infty} a_n = a.$$

Then for any $\varepsilon > 0$, there exists a positive integer N such that

$$|a_n - a| < \varepsilon/2 \text{ for all } n \ge N.$$

For any $m, n \ge N$, one has

$$|a_m - a_n| \le |a_m - a| + |a_n - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, $\{a_n\}$ is a Cauchy sequence. □

Def^n/ (CAUCHY SEQUENCE)

A sequence $\{a_n\}$ in $\mathbb{R}$ is called a Cauchy Sequence if for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $m,n \geq N$ implies $|a_m - a_n| < \varepsilon$.

Th^m (2.4.2) CONVERGENT SEQUENCES ARE CAUCHY SEQUENCES

Proof: Suppose $\{a_n\}$ is a convergent sequence and $a_n \to a$ as $n \to \infty$. Then for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|a_n - a| < \varepsilon/2$ for $n \geq N$. Suppose $m,n \in \mathbb{N}$ and $m,n \geq N$ then,

$$|a_m - a_n| = |a_m - a + a - a_n| \leq |a_m - a| + |a - a_n| = |a_m - a| + |a_n - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\{a_n\}$ is Cauchy Sequence.//

Th^m (2.4.3.) A CAUCHY Sequence is bounded

Proof: If $\{a_n\}$ is cauchy then for $\varepsilon = 1 > 0$ we may select $N \in \mathbb{N}$ for which $|a_m - a_n| < 1$ for all $m,n \geq N$. Then $|a_n - a_N| < 1$ $\forall n \geq N$. Let $M = \max\{|a_1|, \dots, |a_{N-1}|, 1 + |a_N|\}$ and notice, for $n \geq N$,

$$|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N| < 1 + |a_N| \leq M$$

and clearly $|a_n| \leq M$ for $n = 1, 2, \dots, N-1$ thus $|a_n| \leq M$ $\forall n \in \mathbb{N}$.//

$\boxed{\text{Lemma 2.4.4: A Cauchy sequence that has a convergent subsequence is convergent}}$

Proof: Let $\{a_n\}$ be Cauchy. Also suppose $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}_{k=1}^{\infty}$. For any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $m, n \geq N \Rightarrow |a_m - a_n| < \varepsilon/2$. Suppose $a_{n_k} \to a$ as $k \to \infty$. Then, $\exists K \in \mathbb{N}$ s.t. $k \geq K$ implies $|a_{n_k} - a| < \varepsilon/2$.

Thus $\exists n_\ell > N$ s.t. $|a_{n_\ell} - a| < \varepsilon/2$. Consequently, for $n \geq N$,

$$|a_n - a| \leq \underbrace{|a_n - a_{n_\ell}|}_{\text{can make small because of Cauchy Condition}} + \underbrace{|a_{n_\ell} - a|}_{\text{can make small by convergence of subsequence.}} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

if you focus here, it's easy to understand the idea of the proof.

Thus $a_n \to a$ as $n \to \infty$. //

$\boxed{\text{Th}^{\underline{m}}\ (2.4.5)\ \text{Any Cauchy sequence in } \mathbb{R} \text{ is convergent}}$

Proof: If $\{a_n\}$ is Cauchy then by Th$^{\underline{m}}$ 2.4.3 it is bounded. But, then by Bolzano Weierstrass $\{a_n\}$ has a convergent subsequence. Finally, by Lemma 2.4.4. we find $\{a_n\}$ is convergent. //

Remark: the text claimed "Bolzano Weierstrass Th$^{\underline{m}}$ ... is equivalent to the completeness axiom of $\mathbb{R}$". We see this in the above, CAUCHY $\Rightarrow$ CONVERGENT. In contrast, $\exists$ Cauchy seq. inside $\mathbb{Q}$ for which the limit is outside $\mathbb{Q}$. One method to from $\mathbb{R}$ is to take $\mathbb{Q}$ and adjoin the limits of all Cauchy sequences. This process is known as "taking the completion" of $\mathbb{Q}$.

Def^n/ A sequence $\{a_n\}$ is called <u>contractive</u> if $\exists\, k \in [0,1)$ such that $|a_{n+2} - a_{n+1}| \leq k\,|a_{n+1} - a_n|$ $\forall n \in \mathbb{N}$.

Th^m(2.4.7) Every contractive sequence is convergent

Remark 2.4.6: Cauchy iff for every $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $|a_{n+p} - a_n| < \varepsilon$ $\forall n \geq N$ and any $p \in \mathbb{N}$. That is, for each p, $|a_{n+p} - a_n| \longrightarrow 0$ as $n \to \infty$.

Proof: Suppose $\{a_n\}$ is contractive with contraction constant $k \in [0,1)$.

Observe $|a_3 - a_2| \leq k|a_2 - a_1|$ and $|a_4 - a_3| \leq k|a_3 - a_2| \leq k^2|a_2 - a_1|$.

Suppose inductively $|a_{n+2} - a_{n+1}| \leq k^n|a_2 - a_1|$ * we've already shown true for $n = 1, 2$.

Consider $|a_{n+1+2} - a_{n+1+1}| = |a_{n+3} - a_{n+2}| \leq k|a_{n+2} - a_{n+1}| < k\,k^n|a_2 - a_1| = k^{n+1}|a_2 - a_1|$.

Thus * is true $\forall n \in \mathbb{N}$ by PMI. Consider, for all $n, p \in \mathbb{N}$ we have:

$$
\begin{aligned}
|a_{n+p} - a_n| &\leq |a_{n+1} - a_n| + |a_{n+2} - a_{n+1}| + \cdots + |a_{n+p} - a_{n+p-1}| \\
&\leq k^{n-1}|a_2 - a_1| + k^n|a_2 - a_1| + \cdots + k^{n+p-2}|a_2 - a_1| \\
&= (k^{n-1} + k^n + \cdots + k^{n+p-2})|a_2 - a_1| \\
&= k^{n-1}(1 + k + \cdots + k^{p-1})|a_2 - a_1| \\
&\leq k^{n-1}(1 + k + \cdots + k^{p-1} + \cdots)|a_2 - a_1| \\
&\leq \left(\frac{k^{n-1}}{1-k}\right)|a_2 - a_1|.
\end{aligned}
$$

Geometric Series, notice $0 \leq k < 1$

Then, as $n \to \infty$ we find $|a_{n+p} - a_n| \longrightarrow 0$ as $k^{n-1} \longrightarrow 0$ as $n \to \infty$.

$\therefore$ $\{a_n\}$ is Cauchy and hence convergent. //