LECTURE 12: CONNECTED COMPONENTS

Def^n/ Let X be a topological space, a subspace $C \subseteq X$ is called a <u>connected component</u> of X if

(1.) C is connected

(2.) $C \subseteq A$ and A connected $\Rightarrow C = A$ (Maximal connected subset)

$\boxed{E1}$ Suppose $C \subseteq X$ is open, closed, connected and non-empty. Then C is connected component of X. Suppose A is connected and contains C; $C \subseteq A$. But, C is open and closed in $A \Rightarrow A = C$ since the connected set A has no open & closed subsets except $\emptyset$ and A.

<u>Lemma</u>: Let Y be a connected subspace of X and $Y \subset W \subset \overline{Y}$ then W is connected. Moreover, the closure of a connected subspace is connected

<u>Proof</u>: Let $Z \subseteq W$ be non-empty and suppose Z is open & closed in W. Observe $Z \cap Y$ is open and closed in Y. Since Y is dense and Z is open in W we have $Z \cap Y \neq \emptyset$. But, Y is connected $\Rightarrow Y \subseteq Z$. As Y is dense and Z is closed in $W \Rightarrow Z = W$.

Lemma: Let $x_0 \in X$ and $\{Z_i \mid i \in I\}$ a family of connected subspaces in X each of which contains x_0. Then $W = \bigcup_i Z_i$ is connected.

Proof: Suppose towards $\rightarrow\leftarrow$ that $W = A \cup B$ where A, B nonempty open sets with $A \cap B = \emptyset$. Wlog suppose $x_0 \in A$ then $A \cap Z_i \neq \emptyset \;\forall i \in I$.
Suppose $B \cap Z_j \neq \emptyset$ (such j must exist since $B \neq \emptyset$)
then $(A \cap Z_j) \cup (B \cap Z_j) = Z_j$ yet $(A \cap Z_j) \cap (B \cap Z_j) = \emptyset$
thus Z_j is separated by $A \cap Z_j$, $B \cap Z_j$ open, nonempty sets $\Rightarrow Z_j$ is disconnected. $\rightarrow\leftarrow$ Z_j connected. Thus W connected. //

Corollary: Let A, B be connected. If $A \cap B \neq \emptyset$ then $A \cup B$ is connected.

Lemma: Let $x_0 \in X$. Denote $C(x_0) = \bigcup \{Y \mid x_0 \in Y \subseteq X, \; Y \text{ connected}\}$
Then $C(x_0)$ is a connected component of X that contains x_0

Proof: Observe $\{x_0\}$ is connected $\Rightarrow x_0 \in C(x_0)$.
Apply the Lemma above to $\{Y \mid x_0 \in Y \subseteq X, \; Y \text{ connected}\}$ we find $C(x_0)$ is connected. If $A \subseteq X$ is connected and $C(x_0) \subseteq A$ then notice $x_0 \in A$ hence $A \subseteq C(x_0)$ by definition $\therefore$ $C(x_0) = A$.
Therefore, $C(x_0)$ is a connected component of X. //

Def^{n}/ $C(x_0) = \bigcup\{Y \mid x_0 \in Y \subseteq X,\ Y \text{ connected}\}$ is the connected component of X containing x_0

Th^{m}/ Every topological space is the union of its connected components. Each connected component is closed and any point is contained in a unique component.

Proof: since $x \in C(x)$ $\forall x \in X$ we have $\bigcup_{x \in X} C(x) = X$.

$C \subseteq C(x_0)$
$D \subseteq C(x_0)$

Suppose $x_0 \in C$ and $x_0 \in D$ where both C and D are connected components then $x_0 \in C \subseteq C \cup D$ and $x_0 \in D \subseteq C \cup D$ and $C \cup D$ connected by Lemma thus $C = C \cup D = D$. Also, C connect comp. $\Rightarrow \bar{C}$ connected with $C \subseteq \bar{C}$ thus $C = \bar{C}$ which shows C closed.// (applying Lemmas from 1 and 2)

Remark: in general connected components are not open. But, sometimes they are

Lemma: If every point in a topological space X has a connected neighborhood then the connected components of X are open

Proof: Let $C \subseteq X$ be a connected component of X and suppose $x_0 \in C$ with U a connected nbhd of x_0. Since $x_0 \in C \cap U$, the union $C \cup U$ is connected (by the Lemma) and thus $C \cup U \subseteq C \Rightarrow U \subseteq C \Rightarrow C$ is nbhd. of x_0.

Remark: if $F: X \to Y$ is a homeomorphism then F is both open and closed as a map thus F maps connected components to connected components (this seems like a good hwk question) If X and Y have a different # of connected comp. then they're not homeomorphic.

$\boxed{E2}$ $X = \mathbb{R} - \{0\}$ and $Y = \mathbb{R} - \{0,1\}$ are not homeomorphic since $X = (-\infty, 0) \cup (0, \infty)$ where as $Y = (-\infty, 0) \cup (0,1) \cup (1, \infty)$ and I've decomposed X & Y into their connected components.

$\boxed{E3}$

$$GL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) \neq 0\}$$
$$= \underbrace{GL_+(n,\mathbb{R})}_{\det A > 0} \cup \underbrace{GL_-(n,\mathbb{R})}_{\det(A) < 0}$$

The determinant map is polynomial in the coordinates of $\mathbb{R}^{n\times n}$ $\det(A) = \sum \epsilon_{i_1 i_2 \cdots i_n} A_{1i_1} A_{2i_2} \cdots A_{ni_n}$ hence $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a continuous map. Furthermore, $\mathbb{R} - \{0\} = (-\infty, 0) \cup (0,\infty)$ where $(-\infty, 0)$, $(0,\infty)$ are the connected components of $\mathbb{R}^\times = \mathbb{R} - \{0\}$. Note $\det^{-1}(-\infty, 0) = GL_-(n,\mathbb{R})$ and $\det^{-1}(0,\infty) = GL_+(n,\mathbb{R})$ hence $GL_+(n,\mathbb{R})$ and $GL_-(n,\mathbb{R})$ are the connected components of $GL(n,\mathbb{R})$.