LECTURE 13 : COVERS

Def^{n}/ A <u>cover</u> of a set X is a family $\mathcal{A}$ of subsets such that $X = \bigcup \{A \mid A \in \mathcal{A}\}$. The cover is <u>finite</u> if $\mathcal{A}$ is a finite family. The cover is countable if $\mathcal{A}$ is countable. If $\mathcal{A}$ and $\mathcal{B}$ are covers of X and $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{A}$ is a <u>subcover of $\mathcal{B}$</u>

Manetti mentions the map $I \to \mathcal{A}$ for $\mathcal{A} = \{U_i \mid i \in I\}$ given by $i \mapsto U_i$ is onto, but not assumed to be injective.

Def^{n}/ A cover $\mathcal{A}$ of a space X is said to be:

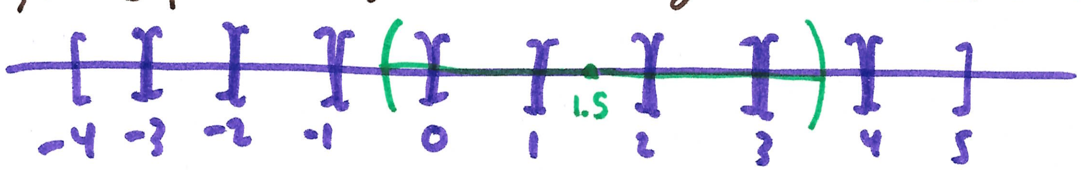
(1.) <u>open</u> if every $A \in \mathcal{A}$ is open,

(2.) <u>closed</u> if every $A \in \mathcal{A}$ is closed,

(3.) <u>locally finite</u> if for every $x \in X$ there exists open $V \subseteq X$ with $x \in V$ and $V \cap A \neq \emptyset$ for at most finitely many $A \in \mathcal{A}$.

$\boxed{E1}$ Any basis of a topology is an open cover.

$\boxed{E2}$ $\{[n, n+1] \mid n \in \mathbb{Z}\}$ is a locally finite closed cover of $\mathbb{R}$.



$A = [0,1], [1,2], [2,3]$

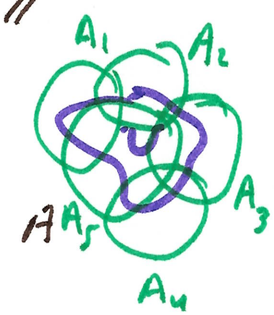
$V = (-0.4, 3.6)$ $\quad V \cap A \neq \emptyset$

Def^{n}/ Let $\mathcal{A}$ be a cover of X. We call $\mathcal{A}$ an <u>identification</u> <u>cover</u> in case $U \subseteq X$ is open iff $U \cap A$ is open $\forall A \in \mathcal{A}$.

$\boxed{E3}$ $\mathcal{A} = \{\{x\} \mid x \in \mathbb{R}\}$ is a closed cover of $\mathbb{R}$ (singleton cover)
However, $\mathcal{A}$ above is <u>not</u> an identification cover, at least not for the Euclidean topology.

<u>Proposition</u>: Let $\mathcal{A}$ be an identification cover of X. A map $f : X \to Y$ is continuous iff for any $A \in \mathcal{A}$ the restriction $f|_A : A \to Y$ is continuous

<u>Proof</u>: Suppose $U \subseteq Y$ is open. If $f|_A : A \to Y$ is continuous $\forall A \in \mathcal{A}$ then $f^{-1}(U) \cap A = (f|_A)^{-1}(U)$ is open in A and noting $\mathcal{A}$ is an identification cover we conclude U is open in X $\therefore$ f is continuous. Conversely, if $f : X \to Y$ is continuous then $f|_A : A \to Y$ is continuous. //



Th^{m}/ Open covers and locally finite closed covers are identification covers.

<u>Proof</u>: Suppose $\mathcal{A}$ is open cover of X and $U \subseteq X$. If $U \cap A$ is open $\forall A \in \mathcal{A}$ in A then $U \cap A$ is open in X as well. So, $U = \bigcup\{A \cap U \mid A \in \mathcal{A}\}$ is open.
Suppose $X = C_1 \cup \dots \cup C_n$ where C_j closed $\forall j = 1, 2, \dots, n$. Let $U \subseteq X$ with $U \cap C_j$ is open in C_j for every j.

Proof continued

$X = C_1 \cup \cdots \cup C_n$ where C_j closed $\forall j = 1, 2, \ldots, n$

Let $U \subseteq X$ such that $U \cap C_j$ is open in C_j $\forall j = 1, 2, \ldots, n$

Write $B = X - U$ and observe $B \cap C_j = C_j - (U \cap C_j)$ is closed in C_j for each $j = 1, 2, \ldots, n$. Thus $B = \bigcup_{i=1}^{n} (B \cap C_i)$ is closed. (finite union of closed is closed)

If $\{C_i \mid i \in I\}$ is locally finite, closed cover, we may select an open cover $\mathcal{A}$ s.t. $\{C_i \cap A \mid i \in I\}$ is closed finite cover of A for any $A \in \mathcal{A}$. Therefore, if $U \cap C_j$ is open in C_j for each j, $U \cap C_j \cap A$ is open in $A \cap C_j$ for all $A \in \mathcal{A}$, and $U \cap A$ is open in A for all $A \in \mathcal{A}$. Thus U is open in X.