LECTURE 13: LIMIT SUPERIOR AND LIMIT INFERIOR

Def^{n}/ Let $\{a_n\}$ be a real sequence. The <u>limit</u> <u>superior</u> of $\{a_n\}$ is denoted $\limsup_{n\to\infty} a_n$ and is defined by:

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}$$

Likewise the <u>limit</u> <u>inferior</u> is defined by:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\}$$

In other words,

$$\limsup_{n\to\infty}(a_n) = \lim_{n\to\infty}\left(\sup\{a_n, a_{n+1}, a_{n+2}, \dots\}\right) \;\&\; \liminf_{n\to\infty}(a_n) = \lim_{n\to\infty}\left(\inf\{a_n, a_{n+1}, a_{n+2}, \dots\}\right)$$

Application: pg 65-66 of Rudin's Principles of Mathematical Analysis 3rd Ed.

<u>Root Test</u>: given $\sum a_n$ put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ then
(1) $\alpha < 1$, $\sum a_n$ conv.
(2.) $\alpha > 1$, $\sum a_n$ div.
(3.) $\alpha = 1$, test not helpful.

<u>Ratio Test</u>: the series $\sum a_n$

(a.) converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ (b.) diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ $\forall n \geq n_0$

Th^{m}/ (3.39, pg. 69) for the power series $\sum c_n z^n$ put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$, $R = \frac{1}{\alpha}$
If $\alpha = 0$ then $R = \infty$, if $\alpha = \infty$ then $R = 0$. Then $\sum c_n z^n$ <u>converges</u> if $|z| < R$. and diverges if $|z| > R$.