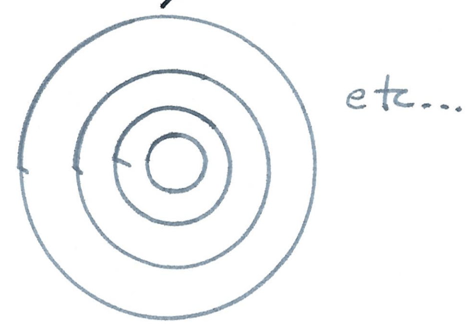
Lecture 14: Compact Spaces

Def^n/ A topological space is <u>compact</u> if any open cover of the space admits a finite subcover. A subspace of a topological space is <u>compact</u> if it is compact for the induced (subspace) topology. Equivalently, a subspace K of X is compact iff for any family $\mathcal{A}$ of open sets in X with $K \subseteq \bigcup\{A \mid A \in \mathcal{A}\}$ there exists finitely many $A_1, A_2, \ldots, A_n \in \mathcal{A}$ for which $K \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$

$\boxed{E1}$ $\mathbb{R}^n$ is not compact. We may prove this assertion by exhibiting an open cover of $\mathbb{R}^n$ for which $\nexists$ a finite subcover

$$\mathbb{R}^n = \bigcup_{m=1}^{\infty} B(0,m)$$



If $\exists$ finite subcover than $\exists M \geq m$ for all m in the subcover so as $B(0,1) \subseteq B(0,2) \subseteq \cdots \subseteq B(0,M)$ we'd have $\mathbb{R}^n = B(0,M)$ which is absurd.

$\boxed{E2}$ Any finite set is compact. Given any open cover we can simply select the singleton cover; $K = \bigcup_{x \in K} \{x\}$. Moreover, a discrete top. space is compact iff it is finite.

Th^{m}/ Let $f : X \to Y$ be a continuous map. If X is compact, the range $f(X)$ is a compact subspace of Y

Proof: Let $\mathcal{A}$ be family of open sets in Y which covers $f(X)$ then $\{f^{-1}(A) \mid A \in \mathcal{A}\}$ is an open cover of $X \Rightarrow \exists$ finite subcover $\{f^{-1}(A_j) \mid j = 1, .., n\}$ for X. Consequently, $f(X) \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$. //

Th^{m}/ The closed interval $[0,1]$ is compact in $\mathbb{R}$

Proof: (Manetti p. 73)

Let $\mathcal{A}$ be family of open subsets in $\mathbb{R}$ which covers $[0,1]$.

Define $X \subseteq [0,\infty)$ to be set of points t s.t. $[0,t] \subseteq \underset{\substack{\text{finitely} \\ \text{many} \\ A \in \mathcal{A}}}{\bigcup} A$. very sneaky step.

Notice $[0,0] = \{0\} \subseteq A$ for some $A \in \mathcal{A}$ as $\mathcal{A}$ covers $[0,1]$ and $0 \in [0,1]$ $\therefore$ $0 \in X$.

Define $b = \text{lub}(X)$. If $b > 1$ then $\exists t \in X$ s.t. $1 \leq t \leq b$ and $[0,1] \subset [0,t]$ is contained in union of finitely many $A \in \mathcal{A}$.

Suppose towards $\rightarrow\leftarrow$ that $b \leq 1$. If $b \leq 1$ then $\exists A \in \mathcal{A}$ s.t. $b \in A$ and since A open $\exists \delta > 0$ for which $(b-\delta, b+\delta) \subseteq A$.

However, by properties of supremum, $\exists t \in X$ s.t. $b - \delta < t \leq b$ then $[0,t]$ is contained in finite union of covering sets; $[0,t] \subseteq A_1 \cup \cdots \cup A_n$. Note, if $0 \leq h < \delta$ then

$$[0, b+h] = [0,t] \cup [t, b+h] \subseteq [0,t] \cup (b-\delta, b+\delta) \subseteq A \cup \underbrace{A_1 \cup \cdots \cup A_n}_{\text{finite union of } A\text{'s in } \mathcal{A}}$$

and thus $b + h \in X$ for any $0 \leq h < \delta$. But, this $\rightarrow\leftarrow$ $b = \sup(X) = \text{lub}(X)$. //

Th$^{\underline{m}}$/ compactness is preserved by homeomorphism

Proof: If X and Y are homeomorphic under the continuous bijection $f : X \longrightarrow Y$ with continuous inverse $f^{-1} : Y \to X$ then the Th$^{\underline{m}}$ from pg. 2 applies as $f(X) = Y$ in this context, hence X compact $\Rightarrow Y$ compact.//

$\boxed{E3}$ $\mathbb{R}$ cannot be homeomorphic to $[0,1]$ since $[0,1]$ is compact whereas $\mathbb{R}$ is not compact (take $n=1$ in $\boxed{E1}$)

Proposition: (1.) Any closed subspace in a compact space is compact. (2.) Moreover, finite unions of compact subspaces are compact.

Proof:

(1.) Suppose Y is closed subspace of compact space X. Suppose $Y \subseteq \bigcup\{A \mid A \in \mathcal{A}\}$ ($\mathcal{A}$ is an open cover of Y). Then $\mathcal{A} \cup (X - Y)$ is an open cover of X $\therefore \exists\ A_1, \dots, A_n \in \mathcal{A}$ for which $X = (X - Y) \cup A_1 \cup \cdots \cup A_n$ hence $Y \subseteq A_1 \cup \cdots \cup A_n$ $\therefore Y$ compact as we've shown an arbitrary open cover of Y contains a finite subcover.

(2.) Let $K_1, \dots, K_n$ be compact subspaces of X and suppose $\mathcal{A}$ is open cover of $K_1 \cup \cdots \cup K_n$. For any $h = 1, 2, \dots, n$, $\exists$ finite $\mathcal{A}_h \subset \mathcal{A}$ that covers K_h (compact) then

$$K = K_1 \cup \cdots \cup K_n \subseteq \bigcup\{A \mid A \in \underbrace{\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n}_{\substack{\text{the finite subcover of } K \text{ is formed by} \\ \text{taking union of the finitely many finite} \\ \text{subcovers of } K_i \text{ for } i=1,2,\dots,n.}}\}.//$$

$\boxed{\text{Corollary: A subspace in } \mathbb{R} \text{ is compact iff it is closed and bounded}}$

Lemma : homeomorphism preserves compactness. (discuss)

Proof : Let $A \subseteq \mathbb{R}$ be closed and bounded. Then $A \subseteq [-a, a]$ for some $a > 0$. But, $[-a, a] \cong [0,1]$ and is thus compact $\Rightarrow$ A compact since it is closed in a compact set (using previous Prop. from 3).
Conversely, if $A \subseteq \mathbb{R}$ is compact then $\{[-n, n] \mid n \in \mathbb{N}\}$ covers A and hence contains a finite subcover $\{[-n_i, n_i] \mid n_i \in \mathbb{N},\ i = 1, 2, .., k\}$ take $N = \max\{n_i \mid i = 1, 2, \dots, k\}$ and note $[-n_i, n_i] \subseteq [-N, N]$ $\forall i = 1, 2, \dots, k$ thus $A \subseteq [-n_1, n_1] \cup \dots \cup [-n_k, n_k] \subset [-N, N] \Rightarrow A$ bounded.
Finally, if $p \notin A$ then $f(x) = \frac{1}{x-p}$ is continuous on $\mathbb{R} - \{p\}$ hence $f(A)$ is compact and thus bounded. Thus $p \notin \bar{A}$ and hence A is closed.

$\boxed{\text{Corollary: Let } X \text{ be compact top. space. Any continuous } f : X \to \mathbb{R} \text{ has a max/min.}}$ $\leftarrow$ extreme value Th$^{\underline{m}}$

Proof : note $f(X)$ is compact in $\mathbb{R}$
$\Rightarrow f(X)$ closed and bounded
$\Rightarrow f(X) \subseteq \mathbb{R}$ has max/min pts. //

Thm/ Consider a closed map $f : X \to Y$ with Y compact. If $f^{-1}\{y\}$ is compact for every $y \in Y$, then X is compact.

Proof: wlog assume $f(X) = Y$, for any $A \subseteq X$ define

$$A' = \{ y \in Y \mid f^{-1}(y) \subseteq A \}$$

Note $Y - A' = f(X - A)$. Then, if A is open $\Rightarrow X - A$ is closed $\Rightarrow f(X - A)$ is closed if f is closed map. Hence, $Y - A'$ closed $\Rightarrow A'$ open.

Let $\mathcal{A}$ be open cover of X. Let $\mathcal{B}$ be the family of finite unions of elements of $\mathcal{A}$. Then $\mathcal{B}' = \{ B' \mid B \in \mathcal{B} \}$ is an open cover of Y. If $y \in Y$ then $f^{-1}\{y\}$ is compact $\Rightarrow \exists A_1, \dots, A_m \in \mathcal{A}$ such that $f^{-1}(y) \subseteq A_1 \cup \dots \cup A_m$ and hence $y \in B'$ for $B = A_1 \cup \dots \cup A_m$.

However as Y is compact, we have finite sequence $B_1, B_2, \dots, B_n \in \mathcal{B}$ such that $Y = B_1' \cup B_2' \cup \dots \cup B_n'$. Then $X = B_1 \cup B_2 \cup \dots \cup B_n$ and as every B_i is finite union of sets in $\mathcal{A}$ we find a finite subcover for $\mathcal{A}$ by collecting such sets. //

Proposition (4.45):

Let $\mathcal{B}$ be a basis of the space X. If every cover of X made by elements of $\mathcal{B}$ has finite subcover, X is compact.

Proof: Manetti p. 75.

Proposition 4.46

Let $K_1 \supseteq K_2 \supseteq \cdots$ be a countable descending chain of nonempty, closed and compact sets. Then

$$\bigcap \{K_n \mid n \in \mathbb{N}\} \neq \emptyset$$

Proof: Manetti p. 75