Lecture 15: Wallace's Theorem

If $A \subseteq X$ and $B \subseteq Y$ then $A \times B \subseteq X \times Y$ and there are a couple ways to topologize $A \times B$

1.) $A \times B$ can be given the subspace topology w.r.t. $X \times Y$

2.) we can take the product topology of the subspace topology of A w.r.t. X with B's subspace top. ind. from Y.

Manetti argues these give the same topology to $A \times B$.

Thm/ (Wallace) Let X, Y be topological spaces, $A \subseteq X$, $B \subseteq Y$ be <u>compact</u> subspaces and $W \subseteq X \times Y$ open set with $A \times B \subseteq W$. Then $\exists$ open $U \subseteq X$ and $V \subseteq Y$ such that

$$A \subseteq U, \quad B \subseteq V \text{ and } U \times V \subseteq W$$

<u>Proof</u>: consider $A = \{a\}$. For any $b \in B$ there exist two open sets $U_b \subseteq X$ and $V_b \subseteq Y$ such that $(a,b) \in U_b \times V_b \subseteq W$. To see why the above is true, note the open family $\{V_b \mid b \in B\}$ covers B hence $\exists\, b_1, \dots, b_n \in B$ s.t. $B \subseteq V_{b_1} \cup \dots \cup V_{b_n}$. Define open sets $U = U_{b_1} \cap \dots \cap U_{b_n}$ and $V = V_{b_1} \cap \dots \cap V_{b_n}$ then,

$$\{a\} \times B \subseteq U \times V \subseteq \bigcup_i U_{b_i} \times V_{b_i} \subseteq W$$

Proof continued:

If A is an arbitrary compact set then by last pg. argument we've shown for any $a \in A$ there are open sets U_a, V_a such that $\{a\} \times B \subseteq U_a \times V_a \subseteq W$. Note that $\{U_a \mid a \in A\}$ is open cover of A $\therefore$ $\exists\, a_1, \dots, a_m \in A$ giving subcover $\{U_{a_1}, \dots, U_{a_m}\}$ for A by compactness of A. then $U = U_{a_1} \cup \cdots \cup U_{a_m}$ and $V = V_{a_1} \cup \cdots \cup V_{a_m}$ give $A \times B \subseteq U \times V \subseteq W$. // (this is all in Manetti)

Corollary 4.48 Any compact subspace in a Hausdorff space is closed

PROOF: Let X be Hausdorff and $K \subseteq X$ compact.
We may show K closed by proving $X - K$ open.
We may show $X - K$ open by demonstrating $x_0 \notin K$ has open $U \subseteq X$ for which $x_0 \in U$ and $U \subseteq X - K$ which means $U \cap K = \emptyset$.
Observe, $\{x_0\} \times K$ does not intersect $\Delta \subseteq X \times X$. But, X Hausdorff, $\{x_0\} \times K \subseteq W = X \times X - \Delta$ *. Apply Wallace's Th$^{\underline{m}}$ to select open $U, V \subseteq X$ for which $\{x_0\} \times K \subseteq U \times V \subseteq W$. Observe $x_0 \in U$ and $U \cap K = \emptyset$ $\therefore$ K closed. //

(*) See Th$^{\underline{m}}$ 3.69, X Hausdorff $\iff$ Δ closed in $X \times X$ $\iff$ $X \times X - \Delta$ is open.

Corollary (4.49): Let X, Y be topological spaces

(1.) if X compact, $\pi_2 : X \times Y \to Y$ by $\pi_2(x,y) = y$ then π_2 is closed map.

(2.) if X and Y are compact then $X \times Y$ is compact.

Proof: (following Manetti pg. 77)

(1.) Suppose $C \subseteq X \times Y$ is closed set.

If $\pi_2(C) = Y$ then Y is closed.

If $\pi_2(C) \neq Y$ then $\exists y \notin \pi_2(C)$. We seek to show $\exists$ nbhd V about y s.t. $V \cap \pi_2(C) = \emptyset$. Consider,

$$X \times \{y\} \subseteq X \times Y - C \leftarrow (C \text{ closed} \Rightarrow X \times Y - C \text{ open.})$$

assumed compact

compact

and applying Wallace's Th$^{\underline{m}}$, $\exists$ open V with $y \in V$ s.t. $(X \times V) \cap C = \emptyset$ thus $V \cap \pi(C) = \emptyset$.

(2.) Recall Th$^{\underline{m}}$ 4.44: closed map $f : \tilde{X} \to \tilde{Y}$ with $\tilde{Y}$ compact. If $f^{-1}\{y\}$ is compact for every $y \in \tilde{Y}$, then $\tilde{X}$ compact.

apply Th$^{\underline{m}}$ 4.44. with $f = \pi_2$ and $\tilde{X} = X \times Y$, $\tilde{Y} = Y$

note $\pi^{-1}\{y\} = X \times \{y\}$ which is compact $\therefore$ $\tilde{X} = X \times Y$ compact. //

$$\psi : X \to X \times \{y\}$$

$$\psi(p) = (p, y) \quad \psi^{-1}(p, y) = p$$

Corollary 4.50: A subset $A \subseteq \mathbb{R}^n$ is compact, in the Euclidean Topology, iff A is closed and bounded.

Proof: If $A \subseteq \mathbb{R}^n$ is closed and bounded then $\exists a > 0$ for which $A \subseteq [-a, a]^n$. However, $[-a, a] \cong [0, 1]$ $\therefore$ $[-a, a]$ compact. Inductively extending Cor 4.49 we see $[-a, a]^n$ is likewise compact. But, a closed subspace of a compact space is compact.

Conversely, if $A \subseteq \mathbb{R}^n$ is compact, the map $d_0 : A \to \mathbb{R}$ given by $d_0(x) = \|x\|$ is continuous with compact domain hence has a maximum. Thus A bounded. But, $\mathbb{R}^n$ is Hausdorff, so Coro. 4.48 provides A is closed. //

$\boxed{\text{E1}}$ Spheres S^n and closed balls D^n are closed and bounded subsets of Euclidean space $\therefore$ S^n and D^n are compact subspaces.

Corollary 4.52: Let $f : X \to Y$ be continuous, X compact and Y Hausdorff. Then f is closed. If f is also bijective then f is a homeomorphism

Proof: Let $A \subseteq X$ be closed. Then A is compact and $f(A)$ is compact in Y Hausdorff $\Rightarrow$ $f(A)$ closed by Cor. 4.48. $\therefore$ f is closed. Then the coro. follows since a continuous and closed bijection is a homeomorphism. //