LECTURE 15: LIMITS OF FUNCTIONS $D \subseteq \mathbb{R}$

Def^{n}/ Let $f: D \to \mathbb{R}$ and $\bar{x}$ be a limit point of D. We say that f has a limit at $\bar{x}$ if there exists a real # ℓ such that for every $\varepsilon > 0$, there exists $\delta > 0$ with $|f(x) - \ell| < \varepsilon$ for all $x \in D$ for which $0 < |x - \bar{x}| < \delta$. In this case we write $\lim_{x \to \bar{x}} f(x) = \ell$.

Remark: $\bar{x}$ need not be a point in the domain D of f.

GOAL: show that $\lim_{x \to 6} (3x+2) = 20$

Ex1| Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x + 2$. Let $\varepsilon > 0$ and choose $\delta = \varepsilon/3$. Suppose $x \in \mathbb{R}$ and $0 < |x-6| < \delta$, then

$$|f(x) - 20| = |3x + 2 - 20| = |3x - 18| = 3|x-6| < 3\delta = 3(\varepsilon/3) = \varepsilon.$$

Thus $\lim_{x \to 6} (3x+2) = 20$.

Ex 2| Let $f(x) = 3x^2 + 2x - 1$ and prove $\lim_{x \to 1} f(x) = 4$.

Scratch work: $|f(x) - f(1)| = |3x^2+2x-1-4| = |3x^2+2x-5| = |(x-1)(3x+5)| < 11|x-1|$

to control $|3x+5|$ need $|x-1| < \delta \le 1 \to -1 < x-1 < 1 \to 0 < x < 2$

$\Rightarrow 0 < 3x < 6$

$\Rightarrow 5 < 3x+5 < 11$

Let $\varepsilon > 0$ and choose $\delta = \min(1, \varepsilon/11)$

and suppose $0 < |x-1| < \delta \Rightarrow 0 < |x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 5 < 3x+5 < 11$

thus $|3x+5| < 11$ and $|x-1| < \varepsilon/11$. Consider then,

$$|f(x) - f(1)| = |3x^2+2x-1-4| = |3x^2+2x-5| = |(3x+5)(x-1)| < 11|x-1| < 11\left(\frac{\varepsilon}{11}\right) = \varepsilon.$$

Thus $\lim_{x \to 1} (3x^2+2x-1) = 4$. //

LECTURE 15

§3.1

LIMITS OF FUNCTIONS
aka CONTINUOUS LIMITS

3. LIMITS AND CONTINUITY

In this chapter, we extend our analysis of limit processes to functions and give the precise definition of continuous function. We derive rigorously two fundamental theorems about continuous functions: the extreme value theorem and the intermediate value theorem.

3.1 LIMITS OF FUNCTIONS

Definition 3.1.1 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. We say that f *has a limit at* $\bar{x}$ if there exists a real number ℓ such that for every $\varepsilon > 0$, there exists $\delta > 0$ with

$$|f(x) - \ell| < \varepsilon$$

for all $x \in D$ for which $0 < |x - \bar{x}| < \delta$. In this case, we write

$$\lim_{x \to \bar{x}} f(x) = \ell.$$

Remark 3.1.1 Note that the limit point $\bar{x}$ in the definition of limit may or may not be an element of the domain D. In any case, the inequality $|f(x) - \ell| < \varepsilon$ need only be satisfied by elements of D.

■ **Example 3.1.1** Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 5x - 7$. We prove that $\lim_{x \to 2} f(x) = 3$. Let $\varepsilon > 0$. First note that $|f(x) - 2| = |5x - 7 - 3| = |5x - 10| = 5|x - 2|$. This suggests the choice $\delta = \varepsilon/5$. Then, if $|x - 2| < \delta$ we have

$$|f(x) - 2| = 5|x - 2| < 5\delta = \varepsilon.$$

■ **Example 3.1.2** Let $f\colon [0,1) \to \mathbb{R}$ be given by $f(x) = x^2 + x$. Let $\bar{x} = 1$ and $\ell = 2$. First note that $|f(x) - \ell| = |x^2 + x - 2| = |x - 1||x + 2|$ and for $x \in [0,1)$, $|x + 2| \le |x| + 2 \le 3$. Now, given $\varepsilon > 0$, choose $\delta = \varepsilon/3$. Then, if $|x - 1| < \delta$ and $x \in [0,1)$, we have

$$|f(x) - \ell| = |x^2 + x - 2| = |x - 1||x + 2| < 3\delta = 3\frac{\varepsilon}{3} = \varepsilon.$$

This shows that $\lim_{x \to 1} f(x) = 2$.

■ **Example 3.1.3** Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. We show that $\lim_{x\to 2} f(x) = 4$. First note that $|f(x) - 4| = |x^2 - 4| = |(x-2)(x+2)| = |x-2||x+2|$. Since the domain is all of $\mathbb{R}$ the expression $|x+2|$ is not bounded and we cannot proceed as in Example 3.1.2. However, we are interested only in values of x close to 2 and, thus, we impose the condition $\delta \le 1$. If $|x-2| < 1$, then $-1 < x-2 < 1$ and, so, $1 < x < 3$. It follows, for such x, that $|x| < 3$ and, hence $|x| + 2 < 5$.

Now, given $\varepsilon > 0$ we choose $\delta = \min\{1, \frac{\varepsilon}{5}\}$. Then, whenever $|x-2| < \delta$ we get

$$|f(x) - 4| = |x-2|x+2| \le |x-2|(|x|+2) < \delta 5 \le \varepsilon.$$

■ **Example 3.1.4** Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \dfrac{3x-5}{x^2+3}$. We prove that $\lim_{x\to 1} f(x) = -\dfrac{1}{2}$. First we look at the expression $|f(x) - (-\frac{1}{2})|$ and try to identify a factor $|x-1|$ (because here $\bar{x} = 1$).

$$\left|f(x) - \left(-\frac{1}{2}\right)\right| = \left|\frac{3x-5}{x^2+3} + \frac{1}{2}\right| = \left|\frac{\overbrace{6x-10+x^2+3}^{x^2+6x-7}}{x^2+3}\right| = \frac{|x-1||x+7|}{2(x^2+3)} \le \frac{1}{6}|x-1||x+7|.$$

0 2

$\bar{x}=1$

$|x-1|<1$

Proceeding as in the previous example, if $|x-1| < 1$ we get $-1 < x-1 < 1$ and, so, $0 < x < 2$. Thus $|x| < 2$ and $|x+7| \le |x| + 2 < 9$. $|x+7|<9$

Now, given $\varepsilon > 0$, we choose $\delta = \min\{1, \frac{2}{3}\varepsilon\}$. It follows that if $|x-1| < \delta$ we get

$$\left|f(x) - (-\frac{1}{2})\right| \le \frac{|x+7|}{6}|x-1| < \frac{9}{6}\delta \le \varepsilon.$$

The following theorem will let us apply our earlier results on limits of sequences to obtain new results on limits of functions.

Theorem 3.1.2 — Sequential Characterization of Limits. Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Then

P
$$\lim_{x\to\bar{x}} f(x) = \ell \tag{3.1}$$

if and only if

Q
$$\lim_{n\to\infty} f(x_n) = \ell \tag{3.2}$$

for every sequence $\{x_n\}$ in D such that $x_n \ne \bar{x}$ for every n and $\{x_n\}$ converges to $\bar{x}$.

$P \Rightarrow Q$

Proof: Suppose (3.1) holds. Let $\{x_n\}$ be a sequence in D with $x_n \ne \bar{x}$ for every n and such that $\{x_n\}$ converges to $\bar{x}$. Given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ whenever $x \in D$ and $0 < |x - \bar{x}| < \delta$. Then there exists $N \in \mathbb{N}$ with $\underbrace{0 < |x_n - \bar{x}|}$ $< \delta$ for all $n \ge N$. For such n, we have

$$|f(x_n) - \ell| < \varepsilon.$$

$x_n = \bar{x} \Rightarrow |x_n - \bar{x}| = 0$

This implies (3.2).

$Q \Rightarrow P$

Conversely, suppose (3.1) is false. Then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$, there exists $x \in D$ with $0 < |x - \bar{x}| < \delta$ and $|f(x) - \ell| \ge \varepsilon_0$. Thus, for every $n \in \mathbb{N}$, there exists $x_n \in D$ with $0 < |x_n - \bar{x}| < \dfrac{1}{n}$ and $|f(x_n) - \ell| \ge \varepsilon_0$. By the squeeze theorem (Theorem 2.1.6), the sequence $\{x_n\}$ converges to $\bar{x}$. Moreover, $x_n \ne \bar{x}$ for every n. This shows that (3.2) is false. It follows that (3.2) implies (3.1) and the proof is complete. □

$\{Q \Rightarrow P\} \Leftrightarrow \{\tilde{P} \Rightarrow \tilde{Q}\}$

$\lim_{x \to \bar{x}} (0) = 0 \qquad f(x) - f(x) = 0$

$0 = \lim_{x \to 0} (f(x) - f(x)) = \lim_{x \to 0} f(x) - \lim_{x \to 0} f(x) = l_1 - l_2 \Rightarrow l_1 = l_2.$

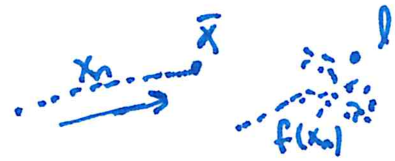
Corollary 3.1.3 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. If f has a limit at $\bar{x}$, then this limit is unique.

Proof: Suppose by contradiction that f has two different limits ℓ_1 and ℓ_2. Let $\{x_n\}$ be a sequence in $D \setminus \{\bar{x}\}$ that converges to $\bar{x}$. By Theorem 3.1.2, the sequence $\{f(x_n)\}$ converges to two different limits ℓ_1 and ℓ_2. This is a contradiction to Theorem 2.1.3. $\square$

The following corollary follows directly from Theorem 3.1.2.

Corollary 3.1.4 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Then f does not have a limit at $\bar{x}$ if and only if there exists a sequence $\{x_n\}$ in D such that $x_n \neq \bar{x}$ for every n, $\{x_n\}$ converges to $\bar{x}$, and $\{f(x_n)\}$ does not converge.

Example 3.1.5 Consider the Dirichlet function $f\colon \mathbb{R} \to \mathbb{R}$ given by



$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{if } x \in \mathbb{Q}^c = \mathbb{R} - \mathbb{Q} \end{cases}$$

Then $\lim_{x \to \bar{x}} f(x)$ does not exist for any $\bar{x} \in \mathbb{R}$. Indeed, fix $\bar{x} \in \mathbb{R}$ and choose two sequences $\{r_n\}$, $\{s_n\}$ converging to $\bar{x}$ such that $r_n \in \mathbb{Q}$ and $s_n \notin \mathbb{Q}$ for all $n \in \mathbb{N}$. Define a new sequence $\{x_n\}$ by

$$x_n = \begin{cases} r_k, & \text{if } n = 2k; \\ s_k, & \text{if } n = 2k - 1. \end{cases}$$

It is clear that $\{x_n\}$ converges to $\bar{x}$. Moreover, since $\{f(r_n)\}$ converges to 1 and $\{f(s_n)\}$ converges to 0, Theorem 2.1.9 implies that the sequence $\{f(x_n)\}$ does not converge. It follows from the sequential characterization of limits that $\lim_{x \to \bar{x}} f(x)$ does not exist.

Theorem 3.1.5 Let $f, g\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Suppose that

$$\lim_{x \to \bar{x}} f(x) = \ell_1, \ \lim_{x \to \bar{x}} g(x) = \ell_2,$$

and that there exists $\delta > 0$ such that

$$f(x) \leq g(x) \text{ for all } x \in B(\bar{x}; \delta) \cap D, x \neq \bar{x}.$$

Then $\ell_1 \leq \ell_2$.

Proof: Let $\{x_n\}$ be a sequence in $B(\bar{x}; \delta) \cap D = (\bar{x} - \delta, \bar{x} + \delta) \cap D$ that converges to $\bar{x}$ and $x_n \neq \bar{x}$ for all n. By Theorem 3.1.2,

$$\lim_{n \to \infty} f(x_n) = \ell_1 \text{ and } \lim_{n \to \infty} g(x_n) = \ell_2.$$

Since $f(x_n) \leq g(x_n)$ for all $n \in \mathbb{N}$, applying Theorem 2.1.5, we obtain $\ell_1 \leq \ell_2$. $\square$

Theorem 3.1.6 Let $f, g\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Suppose

$$\lim_{x \to \bar{x}} f(x) = \ell_1, \ \lim_{x \to \bar{x}} g(x) = \ell_2,$$

and $\ell_1 < \ell_2$. Then there exists $\delta > 0$ such that

$$f(x) < g(x) \text{ for all } x \in B(\bar{x}; \delta) \cap D, x \neq \bar{x}.$$

$l_1 < l_2$

Proof: Choose $\varepsilon > 0$ such that $\ell_1 + \varepsilon < \ell_2 - \varepsilon$ (equivalently, such that $\varepsilon < \frac{\ell_2 - \ell_1}{2}$). Then there exists $\delta > 0$ such that

$$\ell_1 - \varepsilon < f(x) < \ell_1 + \varepsilon \text{ and } \ell_2 - \varepsilon < g(x) < \ell_2 + \varepsilon$$

$|f(x) - l_1| < \varepsilon$ & $|g(x) - l_2| < \varepsilon$

for all $x \in B(\bar{x}; \delta) \cap D, x \neq \bar{x}$. Thus,

$\lim_{x \to \bar{x}} (g(x)) = l_2$

$$f(x) < \ell_1 + \varepsilon \overset{*}{<} \ell_2 - \varepsilon < g(x) \text{ for all } x \in B(\bar{x}; \delta) \cap D, x \neq \bar{x}.$$

The proof is now complete. □

Squeeze Thm

$\boxed{\text{**Theorem 3.1.7** Let } f, g, h \colon D \to \mathbb{R} \text{ and let } \bar{x} \text{ be a limit point of } D.}$

Theorem 3.1.7 Let $f, g, h \colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Suppose there exists $\delta > 0$ such that $f(x) \leq g(x) \leq h(x)$ for all $x \in B(\bar{x}; \delta) \cap D, x \neq \bar{x}$. If $\lim_{x \to \bar{x}} f(x) = \lim_{x \to \bar{x}} h(x) = \ell$, then $\lim_{x \to \bar{x}} g(x) = \ell$.

Proof: The proof is straightforward using Theorem 2.1.6 and Theorem 3.1.2. □

Remark 3.1.8 We will adopt the following convention. When we write $\lim_{x \to \bar{x}} f(x)$ without specifying the domain D of f we will assume that D is the largest subset of $\mathbb{R}$ such that if $x \in D$, then $f(x)$ results in a real number. For example, in

$$\lim_{x \to 2} \frac{1}{x+3}$$

we assume $D = \mathbb{R} \setminus \{-3\}$ and in

$$\lim_{x \to 1} \sqrt{x}$$

we assume $D = [0, \infty)$.

Exercises

3.1.1 Use the definition of limit to prove that

(a) $\lim_{x \to 2} 3x - 7 = -1$.

(b) $\lim_{x \to 3} (x^2 + 1) = 10$.

(c) $\lim_{x \to 1} \dfrac{x+3}{x+1} = 2$.

(d) $\lim_{x \to 0} \sqrt{x} = 0$.

(e) $\lim_{x \to 2} x^3 = 8$.

3.1.2 Prove that the following limits do not exist.

(a) $\lim_{x \to 0} \dfrac{x}{|x|}$.

(b) $\lim_{x \to 0} \cos(1/x)$.

3.1.3 Let $f \colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Prove that if $\lim_{x \to \bar{x}} f(x) = \ell$, then

$$\lim_{x \to \bar{x}} |f(x)| = |\ell|.$$

Give an example to show that the converse is not true in general.