LECTURE 16 : LIMIT THEOREMS

Def^{n}/ Given $f, g : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ and a constant c, the functions $f+g$, fg and cf are functions from D to $\mathbb{R}$ given by pointwise rules,

$$(f+g)(x) = f(x) + g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$(cf)(x) = cf(x).$$

Also $\frac{f}{g} : \tilde{D} \longrightarrow \mathbb{R}$ where $\tilde{D} = \{x \in D \mid g(x) \neq 0\}$ is defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $\forall x \in \tilde{D}$.

Th^{m}/ If $\lim\limits_{x \to \bar{x}} (f(x)) = l_f$ and $\lim\limits_{x \to \bar{x}} (g(x)) = l_g$ then $\lim\limits_{x \to \bar{x}} (f(x) + g(x)) = l_f + l_g$

Proof: Suppose $\lim\limits_{x \to \bar{x}} f(x) = l_f$ and $\lim\limits_{x \to \bar{x}} g(x) = l_g$. Let $\varepsilon > 0$ and choose $\delta_f, \delta_g > 0$ such that $0 < |x - \bar{x}| < \delta_f \Rightarrow |f(x) - l_f| < \varepsilon/2$ and $0 < |x - \bar{x}| < \delta_g \Rightarrow |g(x) - l_g| < \varepsilon/2$.

If $\delta = \min(\delta_f, \delta_g) > 0$ then observe $0 < |x - \bar{x}| < \delta \leq \delta_f, \delta_g$ hence

$$\begin{aligned} |f(x) + g(x) - (l_f + l_g)| &= |f(x) - l_f + g(x) - l_g| \\ &\leq |f(x) - l_f| + |g(x) - l_g| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

$$\lim_{x \to \bar{x}} f+g = \lim_{x \to \bar{x}} f + \lim_{x \to \bar{x}} g$$

Thus $\lim\limits_{x \to \bar{x}} (f(x) + g(x)) = l_f + l_g = \lim\limits_{x \to \bar{x}} f(x) + \lim\limits_{x \to \bar{x}} g(x)$. //

LECTURE 16

§3.2

(2)

3.1.4 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Suppose $f(x) \geq 0$ for all $x \in D$. Prove that if $\lim_{x\to\bar{x}} f(x) = \ell$, then

$$\lim_{x\to\bar{x}} \sqrt{f(x)} = \sqrt{\ell}.$$

3.1.5 Find $\lim_{x\to 0} x\sin(1/x)$.

3.1.6 ▶ Let f be the function given by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \cap [0,1]; \\ 1-x, & \text{if } x \in \mathbb{Q}^c \cap [0,1]. \end{cases}$$

Determine which of the following limits exist. For those that exist find their values.

(a) $\lim_{x\to 1/2} f(x)$.
(b) $\lim_{x\to 0} f(x)$.
(c) $\lim_{x\to 1} f(x)$.

3.2 LIMIT THEOREMS

Here we state and prove various theorems that facilitate the computation of general limits.

Definition 3.2.1 Let $f,g\colon D \to \mathbb{R}$ and let c be a constant. The functions $f+g$, fg, and cf are respectively defined as functions from D to $\mathbb{R}$ by

$$(f+g)(x) = f(x) + g(x),$$
$$(fg)(x) = f(x)g(x),$$
$$(cf)(x) = cf(x)$$

for $x \in D$. Let $\widetilde{D} = \{x \in D : g(x) \neq 0\}$. The function $\frac{f}{g}$ is defined as a function from $\widetilde{D}$ to $\mathbb{R}$ by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

for $x \in \widetilde{D}$.

Theorem 3.2.1 Let $f,g\colon D \to \mathbb{R}$ and let $c \in \mathbb{R}$. Suppose $\bar{x}$ is a limit point of D and

$$\lim_{x\to\bar{x}} f(x) = \ell,\ \lim_{x\to\bar{x}} g(x) = m.$$

Then

(a) $\lim_{x\to\bar{x}} (f+g)(x) = \ell + m$,
(b) $\lim_{x\to\bar{x}} (fg)(x) = \ell m$,
(c) $\lim_{x\to\bar{x}} (cf)(x) = c\ell$,
(d) $\lim_{x\to\bar{x}} \left(\frac{f}{g}\right)(x) = \frac{\ell}{m}$ provided that $m \neq 0$.

③

Proof: Let us first prove (a). Let $\{x_n\}$ be a sequence in D that converges to $\bar{x}$ and $x_n \neq \bar{x}$ for every n. By Theorem 3.1.2,

$$\lim_{n\to\infty} f(x_n) = \ell \text{ and } \lim_{n\to\infty} g(x_n) = m.$$

It follows from Theorem 2.2.1 that

$$\lim_{n\to\infty} (f(x_n) + g(x_n)) = \ell + m.$$

Applying Theorem 3.1.2 again, we get $\lim_{x\to\bar{x}}(f+g)(x) = \ell + m$. The proofs of (b) and (c) are similar.

Let us now show that if $m \neq 0$, then $\bar{x}$ is a limit point of $\widetilde{D}$. Since $\bar{x}$ is a limit point of D, there is a sequence $\{u_k\}$ in D converging to $\bar{x}$ such that $u_k \neq \bar{x}$ for every k. Since $m \neq 0$, it follows from an easy application of Theorem 3.1.6 that there exists $\delta > 0$ with

$$g(x) \neq 0 \text{ whenever } 0 < |x - \bar{x}| < \delta, x \in D.$$

This implies

$$x \in \widetilde{D} \text{ whenever } 0 < |x - \bar{x}| < \delta, x \in D.$$

Then $u_k \in \widetilde{D}$ for all k sufficiently large, and hence $\bar{x}$ is a limit point of $\widetilde{D}$. The rest of the proof of (d) can be completed easily following the proof of (a). □

what is $\widetilde{D}$ Def^n 3.2.1

■ **Example 3.2.1** Consider $f\colon \mathbb{R}\setminus\{-7\} \to \mathbb{R}$ given by $f(x) = \dfrac{x^2+2x-3}{x+7}$. Then, combining all parts of Theorem 3.2.1, we get

$$\lim_{x\to-2} f(x) = \frac{\lim_{x\to-2}(x^2+2x-3)}{\lim_{x\to-2}(x+7)} = \frac{\lim_{x\to-2}x^2 + \lim_{x\to-2}2x - \lim_{x\to-2}3}{\lim_{x\to-2}x + \lim_{x\to-2}7}$$

$$= \frac{(\lim_{x\to-2}x)^2 + 2\lim_{x\to-2}x - \lim_{x\to-2}3}{\lim_{x\to-2}x + \lim_{x\to-2}7} = \frac{(-2)^2+2(-2)-3}{-2+7} = -\frac{3}{5}.$$

■ **Example 3.2.2** We proceed in the same way to compute the following limit.

$$\lim_{x\to0}\frac{1+(2x-1)^2}{x^2+7} = \frac{\lim_{x\to0}1 + \lim_{x\to0}(2x-1)^2}{\lim_{x\to0}x^2 + \lim_{x\to0}7} = \frac{1+1}{0+7} = \frac{2}{7}.$$

■ **Example 3.2.3** We now consider

$$\lim_{x\to-1}\frac{x^2+6x+5}{x+1}. = \lim_{x\to-1}\left[\frac{\cancel{(x+1)}(x+5)}{\cancel{x+1}}\right] = \lim_{x\to-1}(x+5) = \lim_{x\to-1}(x) + \lim_{x\to-1}(5) = -1+5 = \boxed{4}$$

$0 < |x+1| < \delta \Rightarrow x \neq -1$

Since the limit of the denominator is 0 we cannot apply directly part (d) of Theorem 3.2.1. Instead, we first simplify the expression keeping in mind that in the definition of limit we never need to evaluate the expression at the limit point itself. In this case, this means we may assume that $x \neq -1$. For any such x we have

$$\frac{x^2+6x+5}{x+1} = \frac{(x+1)(x+5)}{x+1} = x+5.$$

Therefore,

$$\lim_{x\to-1}\frac{x^2+6x+5}{x+1} = \lim_{x\to-1} x+5 = 4.$$

4

$0<|x-\bar{x}|<\delta \Rightarrow |f(x)-\ell_f|<\varepsilon$

$|f(x)-\ell_f|<\varepsilon$ whenever $0<|x-\bar{x}|<\delta$

Theorem 3.2.2 (Cauchy's criterion) Let $f\colon D\to\mathbb{R}$ and let $\bar{x}$ be a limit point of D. Then f has a limit at $\bar{x}$ if and only if for any $\varepsilon>0$, there exists $\delta>0$ such that

$$|f(r)-f(s)|<\varepsilon \text{ whenever } r,s\in D \text{ and } 0<|r-\bar{x}|<\delta, 0<|s-\bar{x}|<\delta. \tag{3.3}$$

Proof: Suppose $\lim_{x\to\bar{x}} f(x)=\ell$. Given $\varepsilon>0$, there exists $\delta>0$ such that

$$|f(x)-\ell|<\frac{\varepsilon}{2} \text{ whenever } x\in D \text{ and } 0<|x-\bar{x}|<\delta.$$

Thus, for $r,s\in D$ with $0<|r-\bar{x}|<\delta$ and $0<|s-\bar{x}|<\delta$, we have

$$|f(r)-f(s)|\le|f(r)-\ell|+|\ell-f(s)|<\varepsilon.$$

$(f(r)-f(s)=f(r)-\ell+\ell-f(s))$

Let us prove the converse. Fix a sequence $\{u_n\}$ in D such with $\lim_{n\to\infty}u_n=\bar{x}$ and $u_n\neq\bar{x}$ for every n. Given $\varepsilon>0$, there exists $\delta>0$ such that

$$|f(r)-f(s)|<\varepsilon \text{ whenever } r,s\in D \text{ and } 0<|r-\bar{x}|<\delta, 0<|s-\bar{x}|<\delta.$$

Then there exists $N\in\mathbb{N}$ satisfying

$$0<|u_n-\bar{x}|<\delta \text{ for all } n\ge N.$$

($u_n\to\bar{x}$ as $n\to\infty$ was assumed)

This implies

$$|f(u_n)-f(u_m)|<\varepsilon \text{ for all } m,n\ge N.$$

Thus, $\{f(u_n)\}$ is a Cauchy sequence, and hence there exists $\ell\in\mathbb{R}$ such that

$$\lim_{n\to\infty}f(u_n)=\ell.$$

We now prove that f has limit ℓ at $\bar{x}$ using Theorem 3.1.2. Let $\{x_n\}$ be a sequence in D such that $\lim_{n\to\infty}x_n=\bar{x}$ and $x_n\neq\bar{x}$ for every n. By the previous argument, there exists $\ell'\in\mathbb{R}$ such that

$$\lim_{n\to\infty}f(x_n)=\ell'.$$

Fix any $\varepsilon>0$ and let $\delta>0$ satisfy (3.3). There exists $K\in\mathbb{N}$ such that

$$|u_n-\bar{x}|<\delta \text{ and } |x_n-\bar{x}|<\delta$$

for all $n\ge K$. Then $|f(u_n)-f(x_n)|<\varepsilon$ for such n. Letting $n\to\infty$, we have $|\ell-\ell'|\le\varepsilon$. Thus, $\ell=\ell'$ since ε is arbitrary. It now follows from Theorem 3.1.2 that $\lim_{x\to\bar{x}}f(x)=\ell$. $\square$

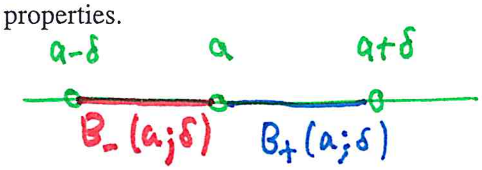
Th^{m} 3.1.2

$\lim_{x\to\bar{x}} f(x)=\ell \iff f(x_n)\to\ell$ for each seq. $\{x_n\}$ in D with $x_n\to\bar{x}$ and $x_n\neq\bar{x}$.

The rest of this section discussed some special limits and their properties.

Definition 3.2.2 Let $a\in\mathbb{R}$ and $\delta>0$. Define

$$B_-(a;\delta)=(a-\delta,a) \text{ and } B_+(a;\delta)=(a,a+\delta).$$



Given a subset A of $\mathbb{R}$, we say that a is a *left limit point of A* if for any $\delta>0$, $B_-(a;\delta)$ contains an infinite number of elements of A. Similarly, a is called a *right limit point of A* if for any $\delta>0$, $B_+(a;\delta)$ contains an infinite number of elements of A.

$B_0(a;\delta)=B_-(a;\delta)\cup B_+(a;\delta)$

It follows from the definition that a is a limit point of A if and only if it is a left limit point of A or it is a right limit point of A.

Definition 3.2.3 (One-sided limits) Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a left limit point of D. We write

$$\lim_{x\to\bar{x}^-} f(x) = \ell$$

if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in B_-(\bar{x};\delta) \Rightarrow |f(x)-\ell| < \varepsilon. \to \lim_{x\to\bar{x}^-} f(x) = \ell$

$$|f(x)-\ell| < \varepsilon \text{ for all } x \in B_-(\bar{x};\delta).$$

$x \in B_+(\bar{x};\delta) \Rightarrow |f(x)-\ell| < \varepsilon \to \lim_{x\to\bar{x}^+} f(x) = \ell$

We say that ℓ is the *left-hand limit of f at $\bar{x}$*. The *right-hand limit of f at $\bar{x}$* can be defined in a similar way and is denoted $\lim_{x\to\bar{x}^+} f(x) = \ell$

■ **Example 3.2.4** Consider the function $f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ given by

$$f(x) = \frac{|x|}{x}.$$

$$\frac{|x|}{x} = \begin{cases} \frac{-x}{x} = -1 & \text{if } x < 0 \\ \frac{x}{x} = 1 & \text{if } x > 0 \end{cases}$$

Let $\bar{x} = 0$. Note first that 0 is a limit point of the set $D = \mathbb{R}\setminus\{0\} \to \mathbb{R}$. Since, for $x > 0$, we have $f(x) = x/x = 1$, we have

$$\lim_{x\to\bar{x}^+} f(x) = \lim_{x\to 0^+} 1 = 1.$$

Similarly, for $x < 0$ we have $f(x) = -x/x = -1$. Therefore,

$$\lim_{x\to\bar{x}^-} f(x) = \lim_{x\to 0^-} -1 = -1.$$

■ **Example 3.2.5** Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x+4, & \text{if } x < -1; \\ x^2-1, & \text{if } x \geq -1. \end{cases} \tag{3.4}$$

We have

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x^2 - 1 = 0,$$

and

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x+4 = 3,$$

The following theorem follows directly from the definition of one-sided limits.

Theorem 3.2.3 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be both a left limit point of D and a right limit point of D. Then

$$\lim_{x\to\bar{x}} f(x) = \ell$$

if and only if

$$\lim_{x\to\bar{x}^+} f(x) = \ell \text{ and } \lim_{x\to\bar{x}^-} f(x) = \ell.$$

■ **Example 3.2.6** It follows from Example 3.2.4 that $\lim_{x\to 0}\frac{|x|}{x}$ does not exists, since the one-sided limits do not agree.

Definition 3.2.4 (monotonicity) Let $f\colon (a,b)\to\mathbb{R}$.

(1) We say that f is *increasing* on (a,b) if, for all $x_1,x_2\in(a,b)$, $x_1<x_2$ implies $f(x_1)\le f(x_2)$.
(2) We say that f is *decreasing* on (a,b) if, for all $x_1,x_2\in(a,b)$, $x_1<x_2$ implies $f(x_1)\ge f(x_2)$.

If f is increasing or decreasing on (a,b), we say that f is monotone on this interval. Strict monotonicity can be defined similarly using strict inequalities: $f(x_1)<f(x_2)$ in (1) and $f(x_1)>f(x_2)$ in (2).

Theorem 3.2.4 Suppose $f\colon (a,b)\to\mathbb{R}$ is increasing on (a,b) and $\bar{x}\in(a,b)$. Then $\lim_{x\to\bar{x}^-}f(x)$ and $\lim_{x\to\bar{x}^+}f(x)$ exist. Moreover,

$$\sup_{a<x<\bar{x}} f(x)=\lim_{x\to\bar{x}^-}f(x)\le f(\bar{x})\le\lim_{x\to\bar{x}^+}f(x)=\inf_{\bar{x}<x<b}f(x).$$

Proof: Since $f(x)\le f(\bar{x})$ for all $x\in(a,\bar{x})$, the set

$a<x<\bar{x} \Rightarrow f(x)\le \underbrace{f(\bar{x})}_{\text{upper bound on}}$

$$\{f(x):x\in(a,\bar{x})\}$$

is nonempty and bounded above. Thus,

$$\ell=\sup_{a<x<\bar{x}}f(x)$$

is a real number. We will show that $\lim_{x\to\bar{x}^-}f(x)=\ell$. For any $\varepsilon>0$, by the definition of the least upper bound, there exists $a<x_1<\bar{x}$ such that

$\ell-\varepsilon \quad f(x_1) \quad \ell \quad f(\bar{x})$

$$\ell-\varepsilon<f(x_1).$$

Let $\delta=\bar{x}-x_1>0$. Using the increasing monotonicity, we get

$x_1<x<\bar{x}$

$$\ell-\varepsilon<f(x_1)\le f(x)\le\ell<\ell+\varepsilon \text{ for all } x\in(x_1,\bar{x})=B_-(\bar{x};\delta).$$

Therefore, $\lim_{x\to\bar{x}^-}f(x)=\ell$. The rest of the proof of the theorem is similar. □

Let

$$B_0(\bar{x};\delta)=B_-(\bar{x};\delta)\cup B_+(\bar{x};\delta)=(\bar{x}-\delta,\bar{x}+\delta)\setminus\{\bar{x}\}.$$

Definition 3.2.5 (infinite limits) Let $f\colon D\to\mathbb{R}$ and let $\bar{x}$ be a limit point of D. We write

$$\lim_{x\to\bar{x}}f(x)=\infty$$

if for every $M\in\mathbb{R}$, there exists $\delta>0$ such that

$$f(x)>M \text{ for all } x\in B_0(\bar{x};\delta)\cap D.$$

$(\bar{x}-\delta,\bar{x})\cup(\bar{x},\bar{x}+\delta)$

Similarly, we write

$$\lim_{x\to\bar{x}}f(x)=-\infty$$

if for every $M\in\mathbb{R}$, there exists $\delta>0$ such that

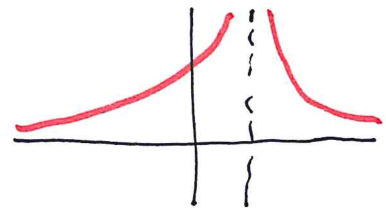
$$f(x)<M \text{ for all } x\in B_0(\bar{x};\delta)\cap D.$$

Infinite limits of functions have similar properties to those of sequences from Chapter 2 (see Definition 2.3.2 and Theorem 2.3.6).

Example 3.2.7 We show that $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$ directly from Definition 3.2.5.

Let $M \in \mathbb{R}$. We want to find $\delta > 0$ that will guarantee $\frac{1}{(x-1)^2} > M$ whenever $0 < |x-1| < \delta$. As in the case of finite limits, we work backwards from $\frac{1}{(x-1)^2} > M$ to an inequality for $|x-1|$. To simplify calculations, note that $|M|+1 > M$. Next note that $\frac{1}{(x-1)^2} > |M|+1$, is equivalent to $\sqrt{\frac{1}{|M|+1}} > |x-1|$.

$y = \frac{1}{(x-1)^2}$



Now, choose δ such that $0 < \delta < \sqrt{\frac{1}{|M|+1}}$. Then, if $0 < |x-1| < \delta$ we have

$$\frac{1}{(x-1)^2} > \frac{1}{\delta^2} > \frac{1}{\frac{1}{|M|+1}} = |M|+1 > M,$$

as desired.

Definition 3.2.6 (limits at infinity) Let $f\colon D \to \mathbb{R}$, where D is not bounded above. We write

$$\lim_{x\to\infty} f(x) = \ell$$

if for every $\varepsilon > 0$, there exists $c \in \mathbb{R}$ such that

$$|f(x) - \ell| < \varepsilon \text{ for all } x > c, x \in D.$$

Let $f\colon D \to \mathbb{R}$, where D is not bounded below. We write

$$\lim_{x\to-\infty} f(x) = \ell$$

if for every $\varepsilon > 0$, there exists $c \in \mathbb{R}$ such that

$$|f(x) - \ell| < \varepsilon \text{ for all } x < c, x \in D.$$

We can also define

$$\lim_{x\to\infty} f(x) = \pm\infty \text{ and } \lim_{x\to-\infty} f(x) = \pm\infty$$

in a similar way.

Example 3.2.8 We prove from the definition that

$$\lim_{x\to-\infty} \frac{3x^2+x}{2x^2+1} = \frac{3}{2}.$$

The approach is similar to that for sequences, with the difference that x need not be an integer.

Let $\varepsilon > 0$. We want to identify c so that

$$\left|\frac{3x^2+x}{2x^2+1} - \frac{3}{2}\right| < \varepsilon, \tag{3.5}$$

for all $x < c$.

Now, $\left|\frac{3x^2+x}{2x^2+1}-\frac{3}{2}\right|=\frac{|2x-3|}{2(2x^2+1)}$. Therefore, simplifying, 3.5 is equivalent to

$$\frac{1}{\varepsilon}<\frac{2(2x^2+1)}{|2x-3|}. \tag{3.6}$$

We first restrict x to be less than 0, so $|2x-3|>3$. Then, since $\frac{4x^2}{3}<\frac{2(2x^2+1)}{|2x-3|}$, 3.6 will be guaranteed if $1/\varepsilon<4x^2/3$ or, equivalently $\sqrt{3/(4\varepsilon)}<|x|$. We set $c<\min\{0,-\sqrt{3/(4\varepsilon)}\}$. Then, if $x<c$, we have $\sqrt{3/(4\varepsilon)}<-x=|x|$. Thus, $1/\varepsilon<\frac{2(2x^2+1)}{|2x-3|}$ and, hence,

$$\left|\frac{3x^2+x}{2x^2+1}-\frac{3}{2}\right|=\frac{|2x-3|}{2(2x^2+1)}<\varepsilon.$$

Exercises

3.2.1 Find the following limits:

(a) $\lim_{x\to 2}\dfrac{3x^2-2x+5}{x-3}$,

(b) $\lim_{x\to -3}\dfrac{x^2+4x+3}{x^2-9}$

3.2.2 Let $f\colon D\to\mathbb{R}$ and let $\bar{x}$ is a limit point of D. Prove that if $\lim_{x\to\bar{x}}f(x)$ exists, then

$$\lim_{x\to\bar{x}}[f(x)]^n=[\lim_{x\to\bar{x}}f(x)]^n, \text{for any } n\in\mathbb{N}.$$

3.2.3 Find the following limits:

(a) $\lim_{x\to 1}\dfrac{\sqrt{x}-1}{x^2-1}$,

(b) $\lim_{x\to 1}\dfrac{x^m-1}{x^n-1}$, where $m,n\in\mathbb{N}$,

(c) $\lim_{x\to 1}\dfrac{\sqrt[n]{x}-1}{\sqrt[m]{x}-1}$, where $m,n\in\mathbb{N}$, $m,n\geq 2$,

(d) $\lim_{x\to 1}\dfrac{\sqrt{x}-\sqrt[3]{x}}{x-1}$.

3.2.4 Find the following limits:

(a) $\lim_{x\to\infty}(\sqrt[3]{x^3+3x^2}-\sqrt{x^2+1})$.

(b) $\lim_{x\to-\infty}(\sqrt[3]{x^3+3x^2}-\sqrt{x^2+1})$.

3.2.5 ⊳ Let $f\colon D\to\mathbb{R}$ and let $\bar{x}$ be a limit point of D. Suppose that

$$|f(x)-f(y)|\leq k|x-y| \text{ for all } x,y\in D\setminus\{\bar{x}\},$$

where $k\geq 0$ is a constant. Prove that $\lim_{x\to\bar{x}}f(x)$ exists.

3.2.6 Determine the one-sided limits $\lim_{x\to 3^+}[x]$ and $\lim_{x\to 3^-}[x]$, where $[x]$ denotes the greatest integer that is less than or equal to x.