LECTURE 16: TOPOLOGICAL GROUPS

Def^{n}/ A <u>topological</u> <u>group</u> is a group G for which multiplication $m : G \times G \longrightarrow G$ and inversion $inv : G \longrightarrow G$ are continuous. Using multiplicative notation $m(g,h) = gh$ and $inv(g) = g^{-1}$.

There are many interesting maps on a topological group. Left/Right multiplication maps give homeomorphisms on G

Def^{n}/ If $h \in G$ then $R_h : G \to G$ is defined by $R_h(x) = xh \ \forall x \in G$. Likewise $L_h : G \to G$ is defined by $L_h(x) = hx \ \forall x \in G$. We call R_h the right multiplication by h map and L_h is left-h-mult. map

Th^{m}/ Let $h \in G$ a topological group, then L_h and R_h are homeomorphisms

<u>Proof</u>: for fixed $h \in G$ the map $x \xmapsto{i_h} (x,h)$ is continuous from $G \to G \times G$ then $R_h = m \circ i_h$ is the composite of continuous maps and is thus cont. Note, $R_h(x) = m(i_h(x)) = m(x,h) = xh$. Observe $h \in G$ and G a group $\Rightarrow \exists h^{-1} \in G$ for which $hh^{-1} = e = h^{-1}h$ where e is the group identity; $ex = x = xe \ \forall x \in G$. Moreover,

$$R_{h^{-1}}(R_h(x)) = R_{h^{-1}}(xh) = (xh)h^{-1} = x$$

and $R_h(R_{h^{-1}}(x)) = xh^{-1}h = x \ \therefore (R_h)^{-1} = R_{h^{-1}}$ and $R_{h^{-1}}$ is continuous on G thus R_h is homeomorphism. Proof for L_h is similar. //

Lemma: For any pair $g, h \in G$ a topological group, there exists a homeomorphism $\varphi : G \to G$ mapping g to h, that is $\varphi(g) = h$

Proof: Let $g, h \in G$ and consider $\varphi = R_{g^{-1}h}$ or $\psi = L_{hg^{-1}}$

SubLemma: $L_a L_b = L_{ab}$ and $R_a R_b = R_{ba}$ $\forall a, b \in G$

PROOF: $(L_a L_b)(x) = L_a(L_b(x)) = L_a(bx) = abx = L_{ab}(x)$ $\forall x \in G$.
$(R_a R_b)(x) = R_a(R_b(x)) = R_a(xb) = xba = R_{ba}(x)$ $\forall x \in G$.

Observe $\varphi = R_{g^{-1}h} = R_h \circ R_{g^{-1}}$ is composite of homeomorphisms thus is a homeomorphism and $\varphi(g) = gg^{-1}h = h$. Likewise $\psi = L_{hg^{-1}} = L_h \circ L_{g^{-1}}$ is homeomorphism as it is comp. of homeomorphisms and $\psi(g) = hg^{-1}g = h$. //

$\boxed{E1}$ Any group G given discrete topology is topological group.

$\boxed{E2}$ Additive groups where $m : G \times G \to G$ is swapped for $+ : G \times G \to G$ and $inv : G \to G$ is swapped for $\underbrace{x \mapsto -x}_{\text{negation}}$ are important examples. For instance, $(\mathbb{R}^n, +)$ and $(\mathbb{C}^n, +)$ with Euclid. Top are Top. Groups.

E2 continued: why is $+ : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ continuous ?

Notice $d_2 : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{R}$ where $\mathbb{X} = \mathbb{R}^n \times \mathbb{R}^n$ can be given by $d_2((x_1, y_1), (x_2, y_2)) = \sqrt{\|x_2 - x_1\|^2 + \|y_2 - y_1\|^2}$

Here d_2 defines distance fnct. on $\mathbb{X} = \mathbb{R}^n \times \mathbb{R}^n$ and $\|x\| = \sqrt{x \cdot x}$ is the usual Euclidean norm on $\mathbb{R}^n$.

Let $F : \mathbb{X} \longrightarrow \mathbb{R}^n$ be defined by $F(x, y) = x + y$.

Suppose $(x_o, y_o) \in \mathbb{X}$ and let $\varepsilon > 0$. Suppose we set $\delta = \varepsilon/2$ and consider $(x, y) \in \mathbb{X}$ such that $(x, y) \in B_\delta(x_o, y_o)$ hence

$$\sqrt{\|x - x_o\|^2 + \|y - y_o\|^2} < \varepsilon/2 \Rightarrow \|x - x_o\|, \|y - y_o\| < \varepsilon/2.$$

Consider the value of F at such (x, y),

$$\begin{aligned} \|F(x,y) - F(x_o, y_o)\| &= \|x + y - (x_o + y_o)\| \\ &= \|x - x_o + y - y_o\| \\ &\leq \|x - x_o\| + \|y - y_o\| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

We find $F^{-1}(B_\varepsilon(F(x_o, y_o))) \subseteq B_\delta(x_o, y_o)$ for arbitrary $(x_o, y_o) \in \mathbb{X}$ thus F is continuous on $\mathbb{X}$ as claimed.

EXAMPLE 3:

$$GL(n,\mathbb{R}) = \{ A \in \mathbb{R}^{n\times n} \mid \det(A) \neq 0 \} = \det^{-1}\big((-\infty,0)\cup(0,\infty)\big)$$

Since $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous we see $GL(n,\mathbb{R})$ is open subset of $\mathbb{R}^{n\times n}$. If we give $GL(n,\mathbb{R})$ the subspace topology w.r.t. to the usual Euclidean topology on $\mathbb{R}^{n\times n}$. In fact,

$$(A,B) \longmapsto AB$$
$$A \longmapsto A^{-1}$$

are both continuous maps on $GL(n,\mathbb{R})$ since:

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$ ← polynomial in coordinates of $\mathbb{R}^{n\times n}$ ∴ continuous.

$$A^{-1} = \underbrace{\frac{1}{\det A}\,(\mathrm{adj}(A))^T}_{\text{classical formula}}$$ ← rational function of coordinates of $\mathbb{R}^{n\times n}$ ∴ continuous.

classical formula
for matrix inverse
in term of the adjoint which
is formed by matrix of cofactors...
Can derive via Cramer's Rule in Linear Algebra.

Lemma 4.57: Let G be topological group with identity e. Then G is Hausdorff iff $\{e\}$ is closed.

Proof: if G is Hausdorff then $\{e\}$ is closed since all singleton sets in a Hausdorff space. (see Lemma 3.67, in fact all finite subsets of Hausdorff space are closed)

Conversely, suppose $\{e\}$ is closed.
Let us study the map $\phi : G \times G \longrightarrow G$ given by $\phi(x,y) = xy^{-1}$.
Note $\phi = L_{\pi_1} \circ (R_{\pi_2})^{-1}$ where $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$
thus ϕ is composite of continuous maps and is thus continuous.
Moreover, $\phi(x,y) = e \iff xy^{-1} = e \iff x = y$
thus $\phi^{-1}\{e\} = \Delta = \{(x,x) \mid x \in G\} \subseteq G \times G$
and $\therefore$ Δ is closed $\Rightarrow$ G is Hausdorff since closed diagonal of a topological space $\Rightarrow$ the space is Hausdorff (Th^{m} 3.69 p. 60 Manetti)

Remark: the existence of left/right multiplication maps on G means what happens at $\{e\}$ happens everywhere. Very similar things happen for a Lie Group which is a group which is also a manifold and the group operations are smooth ...

Examples of Groups

$GL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) \neq 0\}$: general linear grp. over $\mathbb{R}^n$

$SL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) = 1\}$: special linear group in $\mathbb{R}^{n\times n}$

$SO(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid A^T A = I,\ \det A = 1\}$: special orthogonal $n\times n$ matrix group

$GL(n,\mathbb{C}) = \{A \in \mathbb{C}^{n\times n} \mid \det(A) \neq 0\}$: general linear group of complex $n\times n$ matrices.

$SL(n,\mathbb{C}) = \{A \in \mathbb{C}^{n\times n} \mid \det(A) = 1\}$: special linear group of complex $n\times n$ matrices

$U(n,\mathbb{C}) = \{A \in \mathbb{C}^{n\times n} \mid A^\dagger A = I\}$ where $\underbrace{A^\dagger = (\bar{A})^T}_{\text{conjugate transpose of } A \text{ a.k.a. Hermitian adjoint}}$

$SU(n,\mathbb{C}) = U(n,\mathbb{C}) \cap SL(n,\mathbb{C})$

unitary group of $n\times n$ complex matrices

special unitary group of $n\times n$ complex matrices.

- All of the matrix groups above are subspaces of $\mathbb{R}^{n^2}$ or $\mathbb{C}^{n^2}$ (ok, $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$) hence are metrisable $\Rightarrow$ Hausdorff.
- $GL^+(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det A > 0\}$ and $GL(n,\mathbb{C})$ are <u>connected</u> (proof for $GL^+(n,\mathbb{R})$ to follow)

Th$^{\underline{m}}$/ $GL^+(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det A > 0\}$ is connected.

<u>Proof</u>: by induction on n. Base case, $n=1$, notice $GL^+(1,\mathbb{R}) = \{A \in \mathbb{R}^{1\times 1} \mid \det A = A > 0\} = (0,\infty)$ is connected. Suppose $n \in \mathbb{N}$ and $n > 1$ and assume inductively $GL^+(n-1,\mathbb{R})$ connected. Observe $Col_1 : \mathbb{R}^{n\times n} \to \mathbb{R}^n$ defined by $Col_1(A) = \begin{bmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{n1} \end{bmatrix}$.

Note $\mathbb{R}^{n\times n} = \mathbb{R}^n \times \mathbb{R}^{n\times(n-1)}$ then $Col_1 = \pi_1$ for the product space $\mathbb{R}^n \times \mathbb{R}^{n\times(n-1)}$ $\therefore$ Col_1 is continuous and open and hence $Col_1|_{\mathfrak{X}}$ where $\mathfrak{X} = GL^+(n,\mathbb{R})$ is open is likewise an open map. Also, $\underbrace{Col_1(GL^+(n,\mathbb{R})) = \mathbb{R}^n - \{0\}}$

clearly $Col_1(A) = 0 \Rightarrow \det A = 0$

If $V_1 \neq 0$ then continue V_1 to a basis $V_1, V_2, \dots, V_n$ for $\mathbb{R}^n$ and note $A = [V_1|V_2|\cdots|V_n]$ has $Col_1(A) = V_1$ thus Col_1 is onto $\mathbb{R}^n - \{0\}$.

Continuing, Manetti claims (in a slightly different notation) the fibers of $Col_1 : GL^+(n,\mathbb{R}) \to \mathbb{R}^n - \{0\}$ are connected, then with the help of Lemma 4.18 $\Rightarrow GL^+(n,\mathbb{R})$ connected.

Notice,

$$Col_1^{-1}(1,0,\dots,0) = \left\{ A \in GL^+(n,\mathbb{R}) \;\middle|\; A = \left[\begin{smallmatrix}1\\0\\\vdots\\0\end{smallmatrix} \middle| v_2 \middle| v_3 \middle| \cdots \middle| v_n\right] \right\}$$

$$= \{ A = [e_1 | v_2 | \cdots | v_n] \mid \det(A) > 0 \}$$

$$= \{ A = [e_1 | v_2 | \cdots | v_n] \mid \det \underbrace{[\bar{v}_2 | \bar{v}_3 | \cdots | \bar{v}_n]}_{(n-1)\times(n-1)} > 0 \}$$

$$\cong GL^+(n-1,\mathbb{R})$$

$(n-1)\times(n-1)$ matrix formed by cutting of entry one from all n-vectors $v_2, v_3, \dots, v_n$

where $\cong$ is by the map $A \mapsto [\bar{v}_2 | \bar{v}_3 | \cdots | \bar{v}_n]$ which is arguably 1-1 and onto and clearly continuous. Thus, by induction claim $Col_1^{-1}(1,0,\dots,0)$ is connected. Next, we show all fibers of $Col_1 : GL^+(n,\mathbb{R}) \to \mathbb{R}^n - \{0\}$ are homeomorphic. Let $y \in \mathbb{R}^n - \{0\}$ then $Col_1^{-1}(y) \neq \emptyset$ hence let $A \in GL^+(n,\mathbb{R})$ for which $Col_1(A) = y$. Observe $\boxed{Col_1(AB) = A\,Col_1(B)}$ hence

$$L_A(Col_1^{-1}(1,0,\dots,0)) = Col_1^{-1}(y) \Rightarrow Col_1^{-1}(y) \text{ connected since } L_A \text{ homeomorphism.}$$

$$AB = A[B_1 | B_2 | \cdots | B_n] = [AB_1 | AB_2 | \cdots | AB_n]$$

Corollary 4.59: $SL(n,\mathbb{R})$ and $SL(n,\mathbb{C})$ are connected and $GL(n,\mathbb{R})$ has two connected components

Proof: $GL^+(n,\mathbb{R})$ we proved connected and we assume the same can be shown for $GL(n,\mathbb{C})$.

$$\psi : GL^+(n,\mathbb{R}) \longrightarrow SL(n,\mathbb{R})$$

$$\psi(A) = \left[\frac{Col_1(A)}{\det A} \Big| Col_2(A) \Big| \cdots \Big| Col_n(A) \right]$$

Gives a ~~homeomorphism~~ mapping which is continuous and onto. Likewise, $\varphi : GL(n,\mathbb{C}) \longrightarrow SL(n,\mathbb{C})$ given by same trick yields continuous surjection. Thus $SL(n,\mathbb{R})$ and $SL(n,\mathbb{C})$ are continuous images of connected sets $\Rightarrow$ they're connected.
(Th$^{\underline{m}}$ 4.7 if we forget)

Notice $GL(n,\mathbb{R}) = GL^+(n,\mathbb{R}) \cup GL^-(n,\mathbb{R})$ where $GL^-(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det A < 0\}$. Notice $GL^+(n,\mathbb{R}) = \det^{-1}(0,\infty)$ whereas $GL^-(n,\mathbb{R}) = \det^{-1}(-\infty,0)$. Moreover, if $\det B < 0$ then $L_B : GL^+(n,\mathbb{R}) \longrightarrow GL^-(n,\mathbb{R})$ is a homeomorphism $\therefore$ $GL^-(n,\mathbb{R})$ connected, the Cor. follows. //

Lemma 4.60: Let $f : X \to Y$ be continuous and onto from compact X to Y connected and Hausdorff. If all fibers $f^{-1}\{y\}$ are connected then X is connected.

PROOF: by Corollary 4.52, $f : X \to Y$ with X compact and Y Hausdorff gives f closed. Then apply Lemma 4.18 which says $f : X \to Y$ onto continuous map with $f^{-1}\{y\}$ connected $\forall y \in Y$ and f closed then X is connected. //

Proposition: the topological groups $SO(n,\mathbb{R})$, $U(n,\mathbb{C})$ and $SU(n,\mathbb{C})$ are compact and connected.

PROOF: (for $SO(n,\mathbb{R})$) Consider the map $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n} \times \mathbb{R}$ given by $F(A) = (A^TA, \det A)$. We may argue F continuous by virtue of its formula. Moreover,

$$F^{-1}\{(I,1)\} = \{A \in \mathbb{R}^{n\times n} \mid F(A) = (A^TA, \det A) = (I,1)\}$$
$$= \{A \in \mathbb{R}^{n\times n} \mid A^TA = I \text{ and } \det A = 1\} = SO(n,\mathbb{R})$$

Thus $SO(n,\mathbb{R})$ is the continuous inverse image of a $\underbrace{\text{singleton in a Hausdorff}}_{\text{closed set.}}$ space $\therefore$ $SO(n,\mathbb{R})$ is closed. Recall that

$$(A^TA)_{ij} = \sum_k (A^T)_{ik} A_{kj} = \sum A_{ki} A_{kj} = \text{col}_i(A) \cdot \text{col}_j(A) = \delta_{ij}$$

thus $\|\text{col}_i(A)\| = 1$ $\forall i = 1, 2, \dots, n$. Note $\|A\|^2 = \|\text{col}_1(A)\|^2 + \dots + \|\text{col}_n(A)\|^2 = n$ consequently $SO(n,\mathbb{R})$ is bounded. Thus, applying Corollary to $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ we find the closed and bounded $SO(n,\mathbb{R})$ is compact. It remains to show connectedness

<u>Proof continued:</u>

The map $P : SO(n,\mathbb{R}) \longrightarrow S^{n-1}$ defined by $P(A) = Col_1(A)$ is well-stated as $\| Col_1(A) \| = 1$ for all $A \in SO(n,\mathbb{R})$.

A matrix identity for multiplication column-by-column states

$$Col_1(AB) = A\,Col_1(B)$$

thus $P(AB) = A\,P(B)$ $\forall A, B \in SO(n,\mathbb{R})$. Thus, (Manetti claim 2)

$$L_A(P^{-1}(v)) = P^{-1}(Av)$$

Why is this true? For $A \in SO(n,\mathbb{R})$,

$$\begin{aligned}
P^{-1}(Av) &= \{ B \in SO(n,\mathbb{R}) \mid P(B) = Av \} \\
&= \{ B \in SO(n,\mathbb{R}) \mid Col_1(B) = Av \} \\
P^{-1}(v) &= \{ \bar{B} \in SO(n,\mathbb{R}) \mid Col_1(\bar{B}) = v \} \\
L_A(P^{-1}(v)) &= \{ L_A(\bar{B}) \mid Col_1(\bar{B}) = v,\ \bar{B} \in SO(n,\mathbb{R}) \} \\
&= \{ A\bar{B} \mid \underbrace{Col_1(\bar{B}) = v}_{},\ \bar{B} \in SO(n,\mathbb{R}) \}
\end{aligned}$$

But, $A\,col_1(\bar{B}) = Av$ yields $col_1(A\bar{B}) = Av$

$$\begin{aligned}
L_A(P^{-1}(v)) &= \{ A\bar{B} \mid col_1(A\bar{B}) = Av,\ \bar{B} \in SO(n,\mathbb{R}) \} \\
&= \{ B \mid col_1(B) = Av,\ B \in SO(n,\mathbb{R}) \} \\
&= P^{-1}(Av)
\end{aligned}$$

$B = A\bar{B}$

Continued Proof of $SO(n,\mathbb{R})$ connected.

We found $L_A(p^{-1}(v)) = p^{-1}(Av)$ this means the fiber of Av is homeomorphic to the fiber of v for all $v \in S^{n-1}$ and $A \in SO(n,\mathbb{R})$.

① Let $v \in S^{n-1}$ then complete v to orthonormal basis $v, v_2, v_3, \dots, v_n$ for $\mathbb{R}^n$ if $\det[v|v_2|\cdots|v_n] = -1$ then swap v_2 and v_3 to obtain $\det[v_1|v_3|v_2|\cdots|v_n] = 1$. Relable if needed, construct $A = [v|v_2|\cdots|v_n]$ then $A^TA = I$, $\det A = 1$ and $p(A) = \mathrm{Col}_1(A) = v$.

② Consider $p^{-1}(1,0,\dots,0) = \left\{ \left[\begin{array}{c|c} 1 & W \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right] \in SO(n,\mathbb{R}) \right\}$

$$\det\left[\begin{array}{c|c} 1 & W \\ \hline 0 & B \end{array}\right] = \det(1)\det(B) = \det B = 1$$

$$\left[\begin{array}{c|c} 1 & W \\ \hline 0 & B \end{array}\right]^T \left[\begin{array}{c|c} 1 & W \\ \hline 0 & B \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline W^T & B^T \end{array}\right]\left[\begin{array}{c|c} 1 & W \\ \hline 0 & B \end{array}\right] = \left[\begin{array}{c|c} 1 & W \\ \hline W^T & W^TW + B^TB \end{array}\right] = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Thus $W = 0$ and $W^T = 0$ hence $B^TB = I_{n-1}$ thus $B \in SO(n-1,\mathbb{R})$

We find $p^{-1}(1,0,\dots,0) = \left\{ \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & B \end{array}\right] \mid B \in SO(n-1,\mathbb{R}) \right\}$

Continuing proof of $SO(n,\mathbb{R})$ connected

Let $v \in S^{n-1}$ then construct A as in 1 of pg. 12 so $Ae_1 = \mathrm{Col}_1(A) = v$

then notice $\quad L_A(p^{-1}(e_1)) = p^{-1}(Ae_1)$

$$\Rightarrow L_A(SO(n-1,\mathbb{R})) = p^{-1}(v)$$

$\Rightarrow$ all fibers of p are homeomorphic to $SO(n-1,\mathbb{R})$.

- Notice $p : SO(n,\mathbb{R}) \longrightarrow S^{n-1}$ is an onto, continuous map from compact $SO(n,\mathbb{R})$ to S^{n-1} which we know is connected and Hausdorff. If we can prove all fibers of p are connected then Lemma 4.60 gives that the domain of p $(SO(n,\mathbb{R}))$ is connected.

Proceed by induction on n

$SO(1,\mathbb{R}) = \{1\}$ which is connected.

Suppose $SO(n-1,\mathbb{R})$ is connected for some $n \in \mathbb{N}$, $n > 1$ then by arguments of pages $10 \rightarrow 13$ we find $SO(n,\mathbb{R})$ is connected since the induction hypothesis $\Rightarrow$ all fibers of p are connected hence Lemma 4.60 applies and we conclude $SO(n,\mathbb{R})$ is connected.