LECTURE 16: THE GENERAL DERIVATIVE (pg. 180 – 192 in 2020 notes)

Defn/ For map $\vec{F} = (F_1, F_2, \ldots, F_m) : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ we say $\vec{F}$ is differentiable at $\vec{p} \in \mathbb{R}^n$ if $\exists$ linear mapping $\vec{L} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$\lim_{\vec{h} \to 0} \left[\frac{\vec{F}(\vec{p}+\vec{h}) - \vec{F}(\vec{p}) - \vec{L}(\vec{h})}{\|\vec{h}\|} \right] = 0$$

We denote $\vec{L}(\vec{h}) = d\vec{F}_{\vec{p}}(\vec{h})$ and say $d\vec{F}_p$ is the <u>differential</u> of $\vec{F}$ at $\vec{p}$.

Essentially, $\vec{F}(\vec{x}) \cong \underline{\vec{F}(\vec{p}) + d\vec{F}_{\vec{p}}(\vec{x}-\vec{p})}$ gives <u>linearization</u> of $\vec{F}$. Since $d\vec{F}_{\vec{p}}$ is <u>linear transformation</u> it has a <u>standard matrix</u> which is called the <u>Jacobian Matrix</u>. Omitting the point $\vec{p}$,

$$[d\vec{F}] = J_{\vec{F}} = \left[\frac{\partial \vec{F}}{\partial x_1} \middle| \frac{\partial \vec{F}}{\partial x_2} \middle| \cdots \middle| \frac{\partial \vec{F}}{\partial x_n} \right] = \begin{bmatrix} (\nabla F_1)^T \\ (\nabla F_2)^T \\ \vdots \\ (\nabla F_m)^T \end{bmatrix}$$

n = # of domain variables

m = # of range variables

- Can view $J_{\vec{F}}$ either in terms of columns $\frac{\partial \vec{F}}{\partial x_j}$ or rows $(\nabla F_i)^T$

$\xrightarrow[m>1]{n=1}$ $\vec{r}(t)$ has $\frac{d\vec{r}}{dt}$ as $J_{\vec{F}}$ (identify $x_1 = t$, $\vec{F} = \vec{r}$)

$\xrightarrow[n>1]{m=1}$ $f(x_1, \ldots, x_n)$ has $(\nabla f)^T = J_{\vec{F}}$ (identify $F_1 = f$ just one component here)