LECTURE 17

§ 3.3

3.2.7 Find each of the following limits if they exist:

(a) $\lim_{x\to 1^+} \dfrac{x+1}{x-1}$.
(b) $\lim_{x\to 0^+} |x^3 \sin(1/x)|$.
(c) $\lim_{x\to 1}(x-[x])$.

3.2.8 For $a \in \mathbb{R}$, let f be the function given by

$$f(x) = \begin{cases} x^2, & \text{if } x > 1; \\ ax - 1, & \text{if } x \le 1. \end{cases}$$

Find the value of a such that $\lim_{x\to 1} f(x)$ exists.

3.2.9 Determine all values of $\bar{x}$ such that the limit $\lim_{x\to\bar{x}}(1+x-[x])$ exists.

3.2.10 Let $a, b \in \mathbb{R}$ and suppose $f : (a,b) \to \mathbb{R}$ is increasing. Prove the following.

(a) If f is bounded above, then $\lim_{x\to b^-} f(x)$ exists and is a real number.
(b) If f is not bounded above, then $\lim_{x\to b^-} f(x) = \infty$.

State and prove analogous results in case f is bounded below and in case that the domain of f is one of $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$.

3.3 CONTINUITY

LECTURE 17: CONTINUITY

Definition 3.3.1 Let D be a nonempty subset of $\mathbb{R}$ and let $f : D \to \mathbb{R}$ be a function. The function f is said to be *continuous* at $x_0 \in D$ if for any real number $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in D$ and $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| < \varepsilon.$$

x_0 isolated then $\exists \delta > 0$ $B(x_0;\delta) \cap D = \{x_0\}$

$x \in D$ and $|x - x_0| < \delta \Rightarrow x = x_0 \Rightarrow |f(x_0) - f(x)| = 0 < \varepsilon$.

If f is continuous at every point $x \in D$, we say that f is *continuous on D* (or just continuous if no confusion occurs).

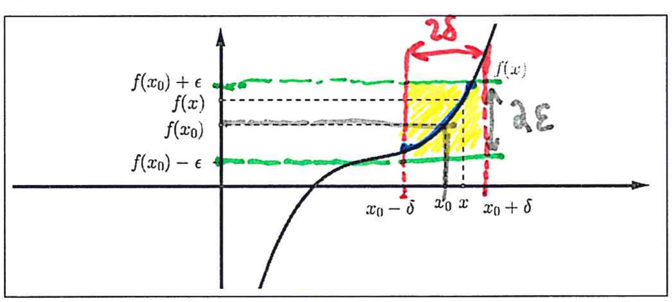

Figure 3.1: Definition of continuity.

Example: $f(x) = x^2 + bx + c$ for $b, c \in \mathbb{R}$, and $x \in \mathbb{R}$.

Claim: f is continuous at $x = x_0$.

Scratch work: get $|x - x_0| < \delta$ to work with, need $|f(x) - f(x_0)| < \varepsilon$.

$$|f(x) - f(x_0)| = |(x^2 + bx + \cancel{c}) - (x_0^2 + bx_0 + \cancel{c})|$$
$$= |x^2 - x_0^2 + b(x - x_0)|$$
$$= |(x - x_0)(x + x_0) + b(x - x_0)|$$
$$= |x - x_0||x + x_0 + b| < \underbrace{\tfrac{\varepsilon}{M} M}_{\text{indicated by work}} = \varepsilon.$$
$$< \delta(|x| + |x_0| + |b|)$$

← true, but I didn't use this... your text tends to build arguments with this.

Ok, $|x - x_0| < \delta \leq 1 \to -1 < x - x_0 < 1$
$$x_0 - 1 < x < 1 + x_0$$
$$2x_0 - 1 < x + x_0 < 2x_0 + 1$$
$$2x_0 - 1 + b < x_0 + b + x < 2x_0 + b + 1$$

Let $M = \max\{|2x_0 + b - 1|, |2x_0 + b + 1|\}$ then $|x - x_0| < 1$ implies $|x_0 + x + b| < M$. So... choose $\delta = \min(1, \varepsilon/M)$

Proof: Let $\varepsilon > 0$ and let $M = \max\{|2x_0 + b - 1|, |2x_0 + b + 1|\}$ and choose $\delta = \min(1, \varepsilon/M)$. (I leave $M = 0$ logic to reader ☺) Suppose $|x - x_0| < \delta$. Then $-1 < x - x_0 < 1 \Rightarrow 2x_0 + b - 1 < x + x_0 + b < 2x_0 + b + 1$ hence $|x + x_0 + b| < M$. Consider then,

$$|f(x) - f(x_0)| = |x^2 + bx + c - (x_0^2 + bx_0 + c)|$$
$$= |x^2 - x_0^2 + bx - bx_0|$$
$$= |(x - x_0)(x + x_0 + b)|$$
$$= |x - x_0||x + x_0 + b| < \delta M \leq \frac{\varepsilon}{M} M = \varepsilon.$$

Thus $f(x)$ is continuous at x_0 and infact, as x_0 was arbitrary we've shown f is continuous on $\mathbb{R}$. //

CALCULUS I $\longrightarrow$ $\left\{ f \text{ continuous at } x_0 \iff \lim_{x \to x_0} f(x) = f(x_0) \right.$

■ **Example 3.3.1** Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x + 7$. Let $x_0 \in \mathbb{R}$ and let $\varepsilon > 0$. Choose $\delta = \varepsilon/3$. Then if $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| = |3x + 7 - (3x_0 + 7)| = |3(x - x_0)| = 3|x - x_0| < 3\delta = \varepsilon.$$

This shows that f is continuous at x_0.

Remark 3.3.1 Note that the above definition of continuity does not mention limits. This allows to include in the definition, points $x_0 \in D$ which are not limit points of D. If x_0 is an isolated point of D, then there is $\delta > 0$ such that $B(x_0; \delta) \cap D = \{x_0\}$. It follows that for $x \in B(x_0; \delta) \cap D$, $|f(x) - f(x_0)| = 0 < \varepsilon$ for any epsilon. Therefore, every function is continuous at an isolated point of its domain.

To study continuity at limit points of D, we have the following theorem which follows directly from the definitions of continuity and limit.

Theorem 3.3.2 Let $f : D \to \mathbb{R}$ and let $x_0 \in D$ be a limit point of D. Then f is continuous at x_0 if and only if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

CALC I Def^{n}.

■ **Example 3.3.2** Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x^2 - 2x + 1$. Fix $x_0 \in \mathbb{R}$. Since, from the results of the previous theorem, we have

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (3x^2 - 2x + 1) = 3x_0^2 - 2x_0 + 1 = f(x_0),$$

it follows that f is continuous at x_0.

The following theorem follows directly from the definition of continuity, Theorem 3.1.2 and Theorem 3.3.2 and we leave its proof as an exercise.

Theorem 3.3.3 Let $f : D \to \mathbb{R}$ and let $x_0 \in D$. Then f is continuous at x_0 if and only if for any sequence $\{x_k\}$ in D that converges to x_0, the sequence $\{f(x_k)\}$ converges to $f(x_0)$.

The proofs of the next two theorems are straightforward using Theorem 3.3.3.

Theorem 3.3.4 Let $f, g : D \to \mathbb{R}$ and let $x_0 \in D$. Suppose f and g are continuous at x_0. Then

(a) $f + g$ and fg are continuous at x_0.

(b) cf is continuous at x_0 for any constant c.

(c) If $g(x_0) \neq 0$, then $\dfrac{f}{g}$ (defined on $\widetilde{D} = \{x \in D : g(x) \neq 0\}$) is continuous at x_0.

Proof: We prove (a) and leave the other parts as an exercise. We will use Theorem 3.3.3. Let $\{x_k\}$ be a sequence in D that converges to x_0. Since f and g are continuous at x_0, by Theorem 3.3.3 we obtain that $\{f(x_k)\}$ converges to $f(x_0)$ and $\{g(x_k)\}$ converges to $g(x_0)$. By Theorem 2.2.1 (a),we get that $\{f(x_k) + g(x_k)\}$ converges to $f(x_0) + g(x_0)$. Therefore,

$$\lim_{k \to \infty} (f + g)(x_k) = \lim_{k \to \infty} f(x_k) + g(x_k) = f(x_0) + g(x_0) = (f + g)(x_0).$$

Since $\{x_k\}$ was arbitrary, using Theorem 3.3.3 again we conclude $f + g$ is continuous at x_0. $\square$

Theorem 3.3.5 Let $f : D \to \mathbb{R}$ and let $g : E \to \mathbb{R}$ with $f(D) \subset E$. If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Th$^{\underline{m}}$ (3.3.5) (COMPOSITION LAW FOR LIMITS)

Let $f : D \longrightarrow \mathbb{R}$ and $g : E \longrightarrow \mathbb{R}$ with $f(D) \subset E$.
If f is continuous at x_o and g is continuous at $f(x_o)$
then $g \circ f$ is continuous at x_o

<u>PROOF</u>: Suppose f continuous at x_o and g continuous at $f(x_o)$.
Let $\varepsilon > 0$ and choose $\delta_g > 0$ for which $|u - f(x_o)| < \delta_g$ and $u \in E$ implies $|g(u) - g(f(x_o))| < \varepsilon$. (we can select such $\delta_g > 0$ by continuity of g at $f(x_o)$)
Likewise, by continuity of f at x_o select $\delta > 0$ for which $|x - x_o| < \delta$ and $x \in D$ $\Rightarrow |f(x) - f(x_o)| < \delta_g$.
Thus for $x \in D$ with $|x - x_o| < \delta$ we find $|f(x) - f(x_o)| < \delta_g$
But, $f(D) \subset E$ hence $f(x) \in E$ and (identifying $u = f(x)$)
we find $|g(f(x)) - g(f(x_o))| < \varepsilon$. Therefore, in summary,

$|x - x_o| < \delta$ and $x \in D$ implies $|(g \circ f)(x) - (g \circ f)(x_o)| < \varepsilon$.

That is, $g \circ f$ is continuous at x_o.

<u>Remark</u>: f continuous at limit point x_o yields: $\lim\limits_{x \to x_o} f(x) = f(x_o)$.
g continuous at limit point $f(x_o)$ yields: $\lim\limits_{u \to f(x_o)} [g(u)] = g(f(x_o))$
Combining these we have the rule:

$$\lim_{x \to x_o} \left(g(f(x)) \right) = g\left(\lim_{x \to x_o} (f(x)) \right)$$

Exercises

3.3.1 Prove, using definition 3.3.1, that each of the following functions is continuous on the given domain:

(a) $f(x) = ax + b$, $a, b \in \mathbb{R}$, on $\mathbb{R}$.

(b) $f(x) = x^2 - 3$ on $\mathbb{R}$.

(c) $f(x) = \sqrt{x}$ on $[0, \infty)$.

(d) $f(x) = \dfrac{1}{x}$ on $\mathbb{R} \setminus \{0\}$.

3.3.2 Determine the values of x at which each function is continuous. The domain of all the functions is $\mathbb{R}$.

(a) $f(x) = \begin{cases} \left|\dfrac{\sin x}{x}\right|, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$

(b) $f(x) = \begin{cases} \dfrac{\sin x}{|x|}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$

(c) $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$

(d) $f(x) = \begin{cases} \cos \dfrac{\pi x}{2}, & \text{if } |x| \leq 1; \\ |x - 1|, & \text{if } |x| > 1. \end{cases}$

(e) $f(x) = \lim\limits_{n \to \infty} \sin \dfrac{\pi}{2(1 + x^{2n})}, \quad x \in \mathbb{R}$.

3.3.3 Let $f \colon \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x^2 + a, & \text{if } x > 2; \\ ax - 1, & \text{if } x \leq 2. \end{cases}$$

Find the value of a such that f is continuous.

3.3.4 Let $f \colon D \to \mathbb{R}$ and let $x_0 \in D$. Prove that if f is continuous at x_0, then $|f|$ is continuous at this point. Is the converse true in general?

3.3.5 Prove Theorem 3.3.3. (*Hint:* treat separately the cases when x_0 is a limit point of D and when it is not.)

3.3.6 Prove parts (b) and (c) of Theorem 3.3.4.

3.3.7 Prove Theorem 3.3.5.

3.3.8 ▶ Explore the continuity of the function f in each case below.