Lecture 17: Exhaustions by Compact Sets

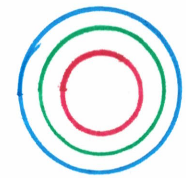
Def^{n}/ An <u>exhaustion by compact sets</u> of a topological space X is a sequence of compact subspaces $\{K_n \mid n \in \mathbb{N}\}$ such that:

(1.) $K_n \subset K_{n+1}^{\circ}$ for any n,

(2.) $\bigcup_n K_n = X$

Note the family of interiors $\{K_n^{\circ}\}$ is an open cover of X, thus for any compact $H \subseteq X$ there exists an $n \in \mathbb{N}$ for which $H \subseteq K_n^{\circ} \subseteq K_n$

$\boxed{E1}$ Let $K_n = \{(x,y) \mid x^2+y^2 \leq n^2\} \subseteq \mathbb{R}^2$.
Then $\{K_n\}$ gives an exhaustion of $\mathbb{R}^2$ by compact sets



$\boxed{E2}$ Following E1 we seek to show $\mathbb{R}^2 \not\approx Y = \mathbb{R}^2 - \{(0,0)\}$. Suppose towards $\rightarrow\leftarrow$ $\exists$ homeomorphism $f : \mathbb{R}^2 \to Y$ then $D_n = f(K_n)$ is an exhaustion of Y by compact sets. Observe, $S^1 \subseteq \mathbb{R}^2$ is compact $\Rightarrow \exists N \in \mathbb{N}$ s.t. $S^1 \subset D_N$. Let $g : Y \to (0,\infty)$ have $g(x,y) = x^2+y^2$ has max $M > 1$ and minimum m with $0 < m < 1$ on the compact set D_N. Thus,

$$\{(x,y) \in Y \mid x^2+y^2 < m\} \subseteq \{(x,y) \in Y - D_N \mid x^2+y^2 < 1\} = A$$
$$\{(x,y) \in Y \mid x^2+y^2 > m\} \subseteq \{(x,y) \in Y - D_N \mid x^2+y^2 > 1\} = B$$

Then $Y - D_N = A \cup B$ where $A \cap B = \emptyset$ and A, B are open. Notice $Y - D_N = f(\mathbb{R}^2 - K_N)$ thus $Y - D_N$ is connected since $\mathbb{R}^2 - K_N$ connected, but we separated $Y - D_N$ with $A \,\&\, B$ $\rightarrow\leftarrow$ $\therefore \mathbb{R}^2 \not\approx \mathbb{R}^2 - \{(0,0)\}$. //

The characteristics of a space at infinity is captured by the one-point compactification of the space:

Def^{n}/ A topological space $\mathbb{X}$ paired with point $\infty \notin \mathbb{X}$ defines $\hat{\mathbb{X}} = \mathbb{X} \cup \{\infty\}$ and give the following as a topology on $\hat{\mathbb{X}}$

$$\mathcal{T} = \{ A \mid A \text{ open in } \mathbb{X} \} \cup \{ \hat{\mathbb{X}} - K \mid K \text{ closed \& compact in } \mathbb{X} \}$$

Th^{m}/ $\hat{\mathbb{X}} = \mathbb{X} \cup \{\infty\}$ with the topology above is a compact space

__Proof:__ begin by noting $\mathcal{T}$ given above does define a topology on $\hat{\mathbb{X}}$ and $i : \mathbb{X} \longrightarrow \hat{\mathbb{X}}$ is an open immersion. Suppose $\{U_i\}$ forms an open cover of $\hat{\mathbb{X}}$. Let $\infty \in U_0$ without loss of generality since $\infty \in \hat{\mathbb{X}} = \mathbb{X} \cup \{\infty\}$ has to be contained somewhere in the cover. Since U_0 is open in $\hat{\mathbb{X}}$ we have $U_0 = \hat{\mathbb{X}} - K$ for some $K \subseteq \mathbb{X}$ which is closed and compact, Notice $\bigcup_{i \neq 0} U_i$ covers $J = \hat{\mathbb{X}} - U_0 = \hat{\mathbb{X}} - (\hat{\mathbb{X}} - K) = K$ which is compact $\therefore \exists$ finite subcover $U_1, \dots, U_n$ for K and we find $\{U_0, U_1, \dots, U_n\}$ is finite subcover of $\{U_i\}$ thus $\hat{\mathbb{X}}$ is compact.//

$\boxed{E3}$ $\mathbb{X} = \mathbb{R}^2 = \mathbb{C}$ then $\hat{\mathbb{X}} = \{\infty\} \cup \mathbb{C}$ has nbhds of ∞ formed by outter annuli. For example: $U = \{\infty\} \cup \{ z \in \mathbb{C} \mid |z| > R \}$

$$= \hat{\mathbb{X}} - D_R(0)$$

OR, $U = \hat{\mathbb{X}} - D_R(z_0)$

PROPOSITION: $\hat{X}$ is Hausdorff $\iff$ X is Hausdorff and every pt. in X has a compact nbhd.

Remark: Manetti's argument is more efficient. See pg. 84.

PROOF: Let $\hat{X} = X \cup \{\infty\}$ where $\infty \notin X$, with $\mathcal{T}$ given on page 2.

$\Leftarrow$) Suppose X Hausdorff and every pt. in X has compact nbhd.
Let $x, y \in \hat{X}$ and consider the possible cases: $(x \neq y)$

(1.) if $x, y \in X$ then $\exists$ open U, V in X and thus $\hat{X}$ for which $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

(2.) if $x \in X$ and $y = \infty$ then $\exists K$ compact containing x, moreover, K° contains x. But, $\hat{X} - K$ is open since K compact in Hausdorff $\Rightarrow$ K closed thus $\infty \in \hat{X} - K$ and $x \in K^\circ$ both open in topology for $\hat{X}$ and $\hat{X} - K \cap K^\circ = \emptyset$

(3.) if $x = \infty$ and $y \in X$ then by symmetry arguement of (2.) applies.

$\Rightarrow$) Suppose $\hat{X}$ is Hausdorff. Let $x, y \in X$ with $x \neq y$ then $x, y \in \hat{X}$ $\Rightarrow$ $\exists U, V$ open in $\hat{X}$ s.t. $x \in U$, $y \in V$ and $U \cap V = \emptyset$. If $U = \hat{X} - K$ for K closed in X then $\tilde{U} = X \cap (\hat{X} - K)$ is open in X and $x \in \tilde{U}$. It follows $\exists \tilde{U}, \tilde{V}$ open in X s.t. $\tilde{U} \cap \tilde{V} = \emptyset$ and $x \in \tilde{U}$, $y \in \tilde{V}$.
If $x \in X$ then since $x \neq \infty$ and $\hat{X}$ Hausdorff, $\exists U, V$ open in $\hat{X}$ with $x \in U$ and $\infty \in V = \hat{X} - K$ for some K closed/compact in X. Then, as $U \cap V = \emptyset$ we find $x \in K \Rightarrow x$ has compact nbhd. //

PROPOSITION: Let $f : X \longrightarrow Y$ be an open immersion of Hausdorff spaces. Then

$$g : Y \longrightarrow \hat{X}, \quad g(y) = \begin{cases} x & \text{if } y = f(x) \\ \infty & \text{if } y \notin f(X) \end{cases}$$

is continuous. In particular, a compact Hausdorff space Y coincides with the one-point compactification of $Y - \{y\}$ for any $y \in Y$.

Proof (Manetti, pg. 84)

Let U be open in $\hat{X}$. If $U \subseteq X$ then $g^{-1}(U) = f(U)$, and so f is open we find $g^{-1}(U) = f(U)$ is open. Then, suppose $U = \hat{X} - K$ for some compact $K \subseteq X$. Observe, $g^{-1}(U) = Y - f(K)$ is open since Cor 4.52 gives f is closed map if we restrict to compact K ($f(K)$ closed $\Rightarrow \underbrace{Y - f(K)}_{\text{open}}$) thus g is continuous.

Finally, suppose Y compact and Hausdorff, set $X = Y - \{y\}$ and $f : X \longrightarrow Y$ the inclusion map. The map g is continuous bijection between compact Hausdorff spaces $\Rightarrow$ g homeomorphism. //