(a) Let $g, h : [0,1] \to \mathbb{R}$ be continuous functions and define

$$f(x) = \begin{cases} g(x), & \text{if } x \in \mathbb{Q} \cap [0,1]; \\ h(x), & \text{if } x \in \mathbb{Q}^c \cap [0,1]. \end{cases}$$

Prove that if $g(a) = h(a)$, for some $a \in [0,1]$, then f is continuous at a.

(b) Let $f \colon [0,1] \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \cap [0,1]; \\ 1-x, & \text{if } x \in \mathbb{Q}^c \cap [0,1]. \end{cases}$$

Find all the points on $[0,1]$ at which the function is continuous.

3.3.9 ▷ Consider the Thomae function defined on $(0,1]$ by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q}, p, q \in \mathbb{N}, \text{where } p \text{ and } q \text{ have no common factors;} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

(a) Prove that for every $\varepsilon > 0$, the set

$$A_\varepsilon = \{x \in (0,1] : f(x) \geq \varepsilon\}$$

is finite.

(b) Prove that f is continuous at every irrational point, and discontinuous at every rational point.

3.3.10 ▷ Consider k distinct points $x_1, x_2, \ldots, x_k \in \mathbb{R}$, $k \geq 1$. Find a function defined on $\mathbb{R}$ that is continuous at each x_i, $i = 1, \ldots, k$, and discontinuous at all other points.

3.3.11 Suppose that f, g are continuous functions on $\mathbb{R}$ and $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

LECTURE 18: PROPERTIES OF CONTINUOUS FUNCTIONS

3.4 PROPERTIES OF CONTINUOUS FUNCTIONS

Recall from Definition 2.6.3 that a subset A of $\mathbb{R}$ is compact if and only if every sequence $\{a_n\}$ in A has a subsequence $\{a_{n_k}\}$ that converges to a point $a \in A$.

Theorem 3.4.1 Let D be a nonempty compact subset of $\mathbb{R}$ and let $f \colon D \to \mathbb{R}$ be a continuous function. Then $f(D)$ is a compact subset of $\mathbb{R}$. In particular, $f(D)$ is closed and bounded.

Proof: Take any sequence $\{y_n\}$ in $f(D)$. Then for each n, there exists $a_n \in D$ such that $y_n = f(a_n)$. Since D is compact, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ and a point $a \in D$ such that

$$\lim_{k\to\infty} a_{n_k} = a \in D.$$

It now follows from Theorem 3.3.3 that

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(a_{n_k}) = f(a) \in f(D).$$

Therefore, $f(D)$ is compact.

The final conclusion follows from Theorem 2.6.5 □

Definition 3.4.1 We say that the function $f\colon D \to \mathbb{R}$ has an *absolute minimum* at $\bar{x} \in D$ if

$$f(x) \geq f(\bar{x}) \text{ for every } x \in D.$$

Similarly, we say that f has an *absolute maximum* at $\bar{x}$ if

$$f(x) \leq f(\bar{x}) \text{ for every } x \in D.$$

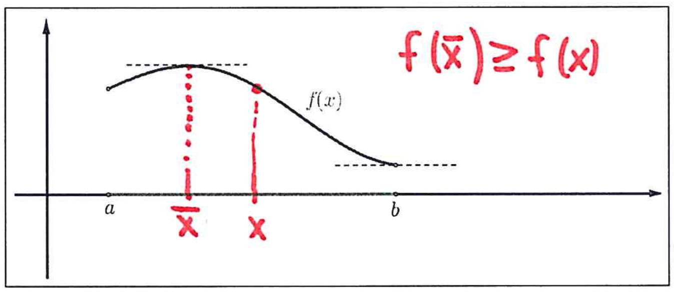


Figure 3.2: Absolute maximum and absolute minimum of f on $[a,b]$.

Theorem 3.4.2 — Extreme Value Theorem. Suppose $f\colon D \to \mathbb{R}$ is continuous and D is a compact set. Then f has an absolute minimum and an absolute maximum on D.

Proof: Since D is compact, $A = f(D)$ is closed and bounded (see Theorem 2.6.5). Let

$$m = \inf A = \inf_{x \in D} f(x).$$

In particular, $m \in \mathbb{R}$. For every $n \in \mathbb{N}$, there exists $a_n \in A$ such that

$$m \leq a_n < m + 1/n.$$

For each n, since $a_n \in A = f(D)$, there exists $x_n \in D$ such that $a_n = f(x_n)$ and, hence,

$$m \leq f(x_n) < m + 1/n.$$

By the compactness of D, there exists an element $\bar{x} \in D$ and a subsequence $\{x_{n_k}\}$ that converges to $\bar{x} \in D$ as $k \to \infty$. Because

$$m \leq f(x_{n_k}) < m + \frac{1}{n_k} \text{ for every } k,$$

by the squeeze theorem (Theorem 2.1.6) we conclude $\lim_{k\to\infty} f(x_{n_k}) = m$. On the other hand, by continuity we have $\lim_{k\to\infty} f(x_{n_k}) = f(\bar{x})$. We conclude that $f(\bar{x}) = m \leq f(x)$ for every $x \in D$. Thus, f has an absolute minimum at $\bar{x}$. The proof is similar for the case of absolute maximum. $\square$

Remark 3.4.3 The proof of Theorem 3.4.2 can be shortened by applying Theorem 2.6.4. However, we have provided a direct proof instead.

Corollary 3.4.4 If $f\colon [a,b] \to \mathbb{R}$ is continuous, then it has an absolute minimum and an absolute maximum on $[a,b]$.

Corollary 3.4.4 is sometimes referred to as the Extreme Value Theorem. It follows immediately from Theorem 3.4.2, and the fact that the interval $[a,b]$ is compact (see Example 2.6.4).

The following result is a basic property of continuous functions that is used in a variety of situations.

Lemma 3.4.5 Let $f\colon D \to \mathbb{R}$ be continuous at $c \in D$. Suppose $f(c) > 0$. Then there exists $\delta > 0$ such that

$$f(x) > 0 \text{ for every } x \in B(c;\delta) \cap D.$$

Proof: Let $\varepsilon = f(c) > 0$. By the continuity of f at c, there exists $\delta > 0$ such that if $x \in D$ and $|x - c| < \delta$, then

$$|f(x) - f(c)| < \varepsilon. \quad \to \; -\varepsilon < f(x) - f(c) < \varepsilon \Rightarrow f(x) < f(c) + \varepsilon$$
$$\Rightarrow f(x) > f(c) - \varepsilon$$

This implies, in particular, that $f(x) > f(c) - \varepsilon = 0$ for every $x \in B(c;\delta) \cap D$. The proof is now complete. □

Remark 3.4.6 An analogous result holds if $f(c) < 0$.

(Bolzano's Thm)

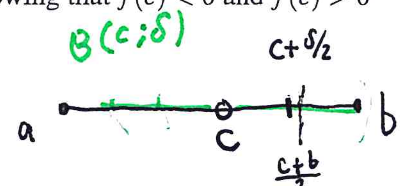
Theorem 3.4.7 Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Suppose $f(a) \cdot f(b) < 0$ (this means either $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$). Then there exists $c \in (a,b)$ such that $f(c) = 0$.

Proof: We prove only the case $f(a) < 0 < f(b)$ (the case $f(a) > 0 > f(b)$ is completely analogous). Define

$$A = \{x \in [a,b] : f(x) \le 0\}. \subseteq [a,b]$$

This set is nonempty since $a \in A$. This set is also bounded since $A \subset [a,b]$. Therefore, $c = \sup A$ exists and $a \le c \le b$. We are going to prove that $f(c) = 0$ by showing that $f(c) < 0$ and $f(c) > 0$ lead to contradictions.

Suppose $f(c) < 0$. Then there exists $\delta > 0$ such that



$$f(x) < 0 \text{ for all } x \in B(c;\delta) \cap [a,b].$$

Because $c < b$ (since $f(b) > 0$), we can find $s \in (c,b)$ such that $f(s) < 0$ (indeed $s = \min\{c + \delta/2, (c+b)/2\}$ will do). This is a contradiction because $s \in A$ and $s > c$.

Suppose $f(c) > 0$. Then there exists $\delta > 0$ such that

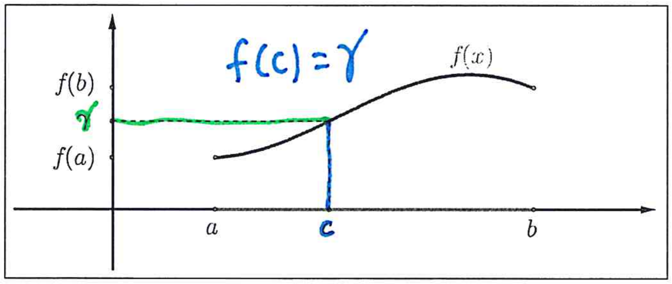
$$f(x) > 0 \text{ for all } x \in B(c;\delta) \cap [a,b].$$

Since $a < c$ (because $f(a) < 0$), there exists $t \in (a,c)$ such that $f(x) > 0$ for all $x \in (t,c)$ (in fact, $t = \max\{c - \delta/2, (a+c)/2\}$ will do). On the other hand, since $t < c = \sup A$, there exists $t' \in A$ with $t < t' \le c$. But then $t < t'$ and $f(t') \le 0$. This is a contradiction. We conclude that $f(c) = 0$. □

Theorem 3.4.8 — Intermediate Value Theorem. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Suppose $f(a) < \gamma < f(b)$. Then there exists a number $c \in (a,b)$ such that $f(c) = \gamma$.

The same conclusion follows if $f(a) > \gamma > f(b)$.

If γ is in-between $f(a)$ and $f(b)$ then $\exists\, c \in (a,b)$ s.t. $f(c) = \gamma$



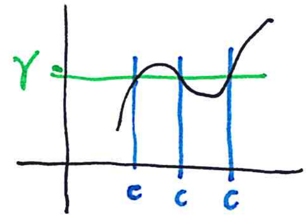


Figure 3.3: Illustration of the Intermediate Value Theorem.

Proof: Define

$$\varphi(x) = f(x) - \gamma,\ x \in [a,b].$$

$f(a) - \gamma < 0$ and $f(b) - \gamma > 0$

$f(a) < \gamma < f(b)$

Then φ is continuous on $[a,b]$. Moreover,

$$\varphi(a)\varphi(b) = [f(a) - \gamma][f(b) - \gamma] < 0.$$

$f(b) < \gamma < f(a)$

$f(b) - \gamma < 0$ and $f(a) - \gamma > 0$

By Theorem 3.4.7, there exists $c \in (a,b)$ such that $\varphi(c) = 0$. This is equivalent to $f(c) = \gamma$. The proof is now complete. □

Corollary 3.4.9 Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Let

$$m = \min\{f(x) : x \in [a,b]\} \text{ and } M = \max\{f(x) : x \in [a,b]\}.$$

Then for every $\gamma \in [m,M]$, there exists $c \in [a,b]$ such that $f(c) = \gamma$.

$f([a,b]) = [m, M]$

Example 3.4.1 We will use the Intermediate Value Theorem to prove that the equation $e^x = -x$ has at least one real solution. We will assume known that the exponential function is continuous on $\mathbb{R}$ and that $e^x < 1$ for $x < 0$.

First define the function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = e^x + x$. Notice that the given equation has a solution x if and only if $f(x) = 0$. Now, the function f is continuous (as the sum of continuous functions). Moreover, note that $f(-1) = e^{-1} + (-1) < 1 - 1 = 0$ and $f(0) = 1 > 0$. We can now apply the Intermediate Value Theorem to the function f on the interval $[-1,0]$ with $\gamma = 0$ to conclude that there is $c \in [-1,0]$ such that $f(c) = 0$. The point c is the desired solution to the original equation.

$\exists\, c \in [-1,0]$

$f(c) = 0$

$e^c + c = 0$

$e^c = -c$

$e^x = -x$

Example 3.4.2 We show now that, given $n \in \mathbb{N}$, every positive real number has a positive n-th root. Let $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ with $a > 0$. First observe that $(1+a)^n \geq 1 + na > a$ (see Exercise 1.3.7). Now consider the function $f\colon [0,\infty) \to \mathbb{R}$ given by $f(x) = x^n$. Since $f(0) = 0$ and $f(1+a) > a$, it follows from the Intermediate Value Theorem that there is $x \in (0, 1+a)$ such that $f(x) = a$. That is, $x^n = a$, as desired. (We show later in Example 4.3.1 that such an x is unique.)

We present below a second proof of Theorem 3.4.8 that does not depend on Theorem 3.4.7, but, instead, relies on the Nested Intervals Theorem (Theorem 2.3.3).

$0 = f(0) < a < f(1+a)$

Second Proof of Theorem 3.4.8: We construct a sequence of nested intervals as follows. Set $a_1 = a$, $b_1 = b$, and let $I_1 = [a,b]$. Let $c_1 = (a+b)/2$. If $f(c_1) = \gamma$, we are done. Otherwise, either

$f(x) = a$

$x^n = a$

$x = \sqrt[n]{a}$

$$f(c_1) > \gamma \quad \text{or}$$
$$f(c_1) < \gamma.$$

In the first case, set $a_2 = a_1$ and $b_1 = c_1$. In the second case, set $a_2 = c_1$ and $b_2 = b_1$. Now set $I_2 = [a_2, b_2]$. Note that in either case,

$$f(a_2) < \gamma < f(b_2).$$

Set $c_2 = (a_2 + b_2)/2$. If $f(c_2) = \gamma$, again we are done. Otherwise, either

$$f(c_2) > \gamma \quad \text{or}$$
$$f(c_2) < \gamma.$$

In the first case, set $a_3 = a_2$ and $b_3 = c_2$. In the second case, set $a_3 = c_2$ and $b_3 = b_2$. Now set $I_3 = [a_3, b_3]$. Note that in either case,

$$f(a_3) < \gamma < f(b_3).$$

Proceeding in this way, either we find some c_{n_0} such that $f(c_{n_0}) = \gamma$ and, hence, the proof is complete, or we construct a sequence of closed bounded intervals $\{I_n\}$ with $I_n = [a_n, b_n]$ such that for all n,
(i) $I_n \supset I_{n+1}$,
(ii) $b_n - a_n = (b-a)/2^{n-1}$, and
(iii) $f(a_n) < \gamma < f(b_n)$.
In this case, we proceed as follows. Condition (ii) implies that $\lim_{n\to\infty}(b_n - a_n) = 0$. By the Nested Intervals Theorem (Theorem 2.3.3, part (b)), there exists $c \in [a,b]$ such that $\bigcap_{n=1}^{\infty} I_n = \{c\}$. Moreover, as we see from the proof of that theorem, $a_n \to c$ and $b_n \to c$ as $n \to \infty$.

By the continuity of f, we get

$$\lim_{n\to\infty} f(a_n) = f(c) \quad \text{and}$$
$$\lim_{n\to\infty} f(b_n) = f(c).$$

Since $f(a_n) < \gamma < f(b_n)$ for all n, condition (iii) above and Theorem 2.1.5 give

$$f(c) \le \gamma \quad \text{and}$$
$$f(c) \ge \gamma.$$

It follows that $f(c) = \gamma$. Note that, since $f(a) < \gamma < f(b)$, then $c \in (a,b)$. The proof is now complete. □

Now we are going to discuss the continuity of the inverse function. For a function $f\colon D \to E$, where E is a subset of $\mathbb{R}$, we can define the new function $f\colon D \to \mathbb{R}$ by the same function notation. The function $f\colon D \to E$ is said to be continuous at a point $\bar{x} \in D$ if the corresponding function $f\colon D \to \mathbb{R}$ is continuous at $\bar{x}$.

Theorem 3.4.10 Let $f\colon [a,b] \to \mathbb{R}$ be strictly increasing and continuous on $[a,b]$. Let $c = f(a)$ and $d = f(b)$. Then f is one-to-one, $f([a,b]) = [c,d]$, and the inverse function f^{-1} defined on $[c,d]$ by

$$f^{-1}(f(x)) = x \text{ where } x \in [a,b],$$

is a continuous function from $[c,d]$ onto $[a,b]$.

$f([a,b]) = [f(a), f(b)]$
$f(a) < f(x) < f(b) \quad \forall x \in (a,b)$

Proof: The first two assertions follow from the monotonicity of f and the Intermediate Value Theorem (see also Corollary 3.4.9). We will prove that f^{-1} is continuous on $[c,d]$. Fix any $\bar{y} \in [c,d]$ and fix any sequence $\{y_k\}$ in $[c,d]$ that converges to $\bar{y}$. Let $\bar{x} \in [a,b]$ and $x_k \in [a,b]$ be such that

$$f(\bar{x}) = \bar{y} \text{ and } f(x_k) = y_k \text{ for every } k.$$

Then $f^{-1}(\bar{y}) = \bar{x}$ and $f^{-1}(y_k) = x_k$ for every k. Suppose by contradiction that $\{x_k\}$ does not converge to $\bar{x}$. Then there exist $\varepsilon_0 > 0$ and a subsequence $\{x_{k_\ell}\}$ of $\{x_k\}$ such that

$$|x_{k_\ell} - \bar{x}| \geq \varepsilon_0 \text{ for every } \ell. \tag{3.7}$$

Since the sequence $\{x_{k_\ell}\}$ is bounded, it has a further subsequence that converges to $x_0 \in [a,b]$. To simplify the notation, we will again call the new subsequence $\{x_{k_\ell}\}$. Taking limits in (3.7), we get

$$|x_0 - \bar{x}| \geq \varepsilon_0 > 0. \tag{3.8}$$

On the other hand, by the continuity of f, $\{f(x_{k_\ell})\}$ converges to $f(x_0)$. Since $f(x_{k_\ell}) = y_{k_\ell} \to \bar{y}$ as $\ell \to \infty$, it follows that $f(x_0) = \bar{y} = f(\bar{x})$. This implies $x_0 = \bar{x}$, which contradicts (3.8). $\square$

Remark 3.4.11 A similar result holds if the domain of f is the open interval (a,b) with some additional considerations. If $f\colon (a,b) \to \mathbb{R}$ is increasing and bounded, following the argument in Theorem 3.2.4 we can show that both $\lim_{x\to a^+} f(x) = c$ and $\lim_{x\to b^-} f(x) = d$ exist in $\mathbb{R}$ (see Exercise 3.2.10). Using the Intermediate Value Theorem we obtain that $f((a,b)) = (c,d)$. We can now proceed as in the previous theorem to show that f has a continuous inverse from (c,d) to (a,b).

If : $(a,b) \to \mathbb{R}$ is increasing, continuous, bounded below, but not bounded above, then $\lim_{x\to a^+} f(x) = c \in \mathbb{R}$, but $\lim_{x\to b^-} f(x) = \infty$ (again see Exercise 3.2.10). In this case we can show using the Intermediate Value Theorem that $f((a,b)) = (c,\infty)$ and we can proceed as above to prove that f has a continuous inverse from (c,∞) to (a,b).

The other possibilities lead to similar results. $f([a,b]) = [f(b), f(a)]$

A similar theorem can be proved for strictly decreasing functions.

Exercises

3.4.1 Let $f\colon D \to \mathbb{R}$ be continuous at $c \in D$ and let $\gamma \in \mathbb{R}$. Suppose $f(c) > \gamma$. Prove that there exists $\delta > 0$ such that

$$f(x) > \gamma \text{ for every } x \in B(c;\delta) \cap D.$$

3.4.2 Let f,g be continuous functions on $[a,b]$. Suppose $f(a) < g(a)$ and $f(b) > g(b)$. Prove that there exists $x_0 \in (a,b)$ such that $f(x_0) = g(x_0)$.

3.4.3 Prove that the equation $\cos x = x$ has at least on solution in $\mathbb{R}$. (Assume known that the function $\cos x$ is continuous.)

3.4.4 Prove that the equation $x^2 - 2 = \cos(x+1)$ has at least two real solutions. (Assume known that the function $\cos x$ is continuous.)

3.4.5 Let $f\colon [a,b] \to [a,b]$ be a continuous function.