LECTURE 18: QUOTIENT TOPOLOGY

"quotient map"

Def^{n}/ A continuous and onto map $f : X \to Y$ is called an <u>identification</u> if the open sets of Y are precisely the subsets $A \subseteq Y$ s.t. $f^{-1}(A)$ is open in X

This is stronger than continuity. Recall, $f : X \to Y$ continuous if for each open $V \subseteq Y$ we have $f^{-1}(V)$ is open in X. It is on the other hand, not as strong as the condition f is an open map. In fact, there exist identifications which are not open maps. (<u>for</u> <u>example</u>, #2 on p. 143 of Munkres Topology a 1st Course, 1st Ed.)

Remark: identifications can be characterized by closed sets just the same; if the closed sets of Y are precisely the subsets $C \subseteq Y$ s.t $f^{-1}(C)$ is closed in X then f is an identification

Def^{n}/ For $f : X \to Y$ continuous, $A \subseteq X$ is f-saturated whenever $x \in A$, $y \in X$ and $f(x) = f(y)$ imply $y \in A$. That is, f-saturated subsets of X are precisely those sets which arise as inverse images under f; $A = f^{-1}(B)$ for some $B \subseteq Y$ means A is f-saturated

Manetti says on p. 87, $\approx$"saying f is an identification amounts to saying the open sets in Y are exactly the images $f(A)$ of f-saturated sets A"

$\boxed{E1}$ Let $\mathcal{A}$ be cover of the space X and consider the <u>disjoint</u> <u>union</u> $\coprod \{A \mid A \in \mathcal{A}\} = Y$. Then $f : Y \to X$ given by $f(x) = x$ $\forall x \in Y$ is a continuous surjection. For f to be an <u>identification</u> we need the open sets of X are precisely $A \subseteq X$ such that $f^{-1}(A)$ is open in Y. Notice $f^{-1}(A) = A$. Manetti claims f is an identification iff $\mathcal{A}$ is an <u>identification cover</u>. (see p. 72 for def^{n} of ident. cover)

Def^{n}/ A closed identification is an identification and a closed map. An open identification is an identification and an open map.

Lemma: Suppose $f : X \to Y$ is a continuous surjection. If f is closed, it is a closed identification. Likewise, if f is open, it is an open identification

Proof: f surjective implies $f(f^{-1}(A)) = A$ for any $A \subseteq Y$.
Suppose f is open and $A \subseteq Y$ is such that $f^{-1}(A)$ is open then $A = f(f^{-1}(A))$ is open $\therefore$ f is an identification. Likewise, if f is closed and $C \subseteq Y$ is such that $f^{-1}(C)$ is closed then $C = f(f^{-1}(C))$ is closed $\Rightarrow$ f is an identification. //

$\boxed{E2}$ Consider $f : [0, 2\pi] \longrightarrow S^1 = \{ (x,y) \mid x^2+y^2 = 1 \}$
$f(t) = (\cos t, \sin t)$

Remark: $[0,1)$ open in $[0,2\pi]$ but $f[0,1)$ not open in S^1 thus f not open, yet it is an identification.

Since $[0,2\pi]$ is closed and bounded, it's compact and S^1 is Hausdorff thus f is a closed map. (Cor. 4.52). Moreover, f is onto $\therefore$ f is closed identification.

$\boxed{E3}$ $Y = ((0,\infty) \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) \subseteq \mathbb{R} \times \mathbb{R}$ and let $h = \pi_1 |_Y$ then we can show h is neither open nor closed, but it is a quotient map.
Munkres gives hint: $h^{-1}(U) \cap (\mathbb{R} \times \{0\}) = U \times \{0\}$

$$h = \pi_1|_Y : ((0,\infty) \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) \longrightarrow \mathbb{R}$$

continuous ✓
surjective ✓
⋮
(left to reader)

Lemma (5.6) Universal Property of Quotient Maps)

Let $f : X \to Y$ be an identification and $g : X \to Z$ a continuous map. There exists a continuous map $h : Y \to Z$ such that $g = h \circ f$ iff g is constant on the fibers of f

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

Proof: Assume $f : X \to Y$ is quotient map and $g : X \to Z$ is continuous.

$\Rightarrow$) Suppose $\exists\, h : Y \to Z$ continuous with $g = h \circ f$. Let $y \in Y$ and suppose $x_1, x_2 \in f^{-1}(y)$ then $f(x_1) = y = f(x_2)$. Observe

$$g(x_1) = h(f(x_1)) = h(f(x_2)) = g(x_2)$$ $\therefore$ g is constant on fibers of f. //

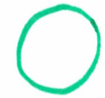
$\Leftarrow$) Suppose g is constant on the fibers of f then if $x_1, x_2 \in f^{-1}(y)$ for some $y \in Y$ then $g(x_1) = g(x_2)$.

Suppose $y \in Y$ where $y = f(x)$ for some $x \in X$, define $h(y) = g(x)$ notice if $y = f(\bar{x})$ then $x, \bar{x} \in f^{-1}(y)$ thus $g(x) = g(\bar{x})$ and hence h is well-defined. Suppose $U \subseteq Z$ is open and notice

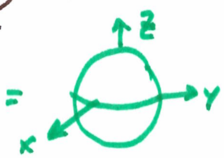
$$g = h \circ f \Rightarrow g^{-1}(U) = f^{-1}(h^{-1}(U))$$ and $g^{-1}(U)$ is open by cont. of g

whence $h^{-1}(U)$ is open as f is an identification. //

E4 4

$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \subseteq \mathbb{R}^n$ $\quad D^1 = [-1,1]$ $\quad S^1 =$ 

$S^n = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\|^2 + \|y\|^2 = \|x\|^2 + y^2 = 1\} \subseteq \mathbb{R}^{n+1}$

$f : D^n \longrightarrow S^n$ $\quad D^2 =$  $\quad S^2 =$ 

$f(x) = \left(2x\sqrt{1-\|x\|^2},\ 2\|x\|^2 - 1\right)$ $\leftarrow$ continuous since f-la behaves nicely for $\|x\|^2 \leq 1$.

<u>Onto?</u> Consider $(a,b) \in S^n$ and seek $x \in D^n$ for which $f(x) = (a,b)$

$a = 2x\sqrt{1-\|x\|^2}$ (a is in the x-direction)

$b = 2\|x\|^2 - 1 \Rightarrow \|x\|^2 = \frac{b+1}{2}$ $\therefore \|x\| = \sqrt{\frac{b+1}{2}}$

Then use $x = \frac{a}{\|a\|}\sqrt{\frac{b+1}{2}}$ where $\|a\|^2 + b^2 = 1 \rightarrow \|a\| = \sqrt{1-b^2}$

Check it, $f\left(\frac{a}{\|a\|}\sqrt{\frac{b+1}{2}}\right) = \left(2\frac{a}{\|a\|}\sqrt{\frac{b+1}{2}}\sqrt{1-\frac{b+1}{2}},\ 2\left(\frac{b+1}{2}\right) - 1\right)$

$= \left(2\frac{a}{\|a\|}\sqrt{\frac{b+1}{2}}\sqrt{\frac{1-b}{2}},\ b\right)$

$= \left(a\frac{1}{\|a\|}\sqrt{1-b^2},\ b\right)$

$= (a,b).$

$\tilde{f}:$ 

$\tilde{f}:$ 

$\boxed{\begin{array}{l} f(0) = (0,-1) \\ f(\partial D^n) = \{(0,1)\} \\ \partial D^n : \|x\| = 1 \end{array}}$

$\tilde{f} : \{x \in D^n \mid \|x\| < 1\} \longrightarrow \{(x,y) \in S^n \mid y < 1\}$ is a homeomorphism

$f(\partial D^n) = \{(0,1)\} \in S^n$

"$\tilde{f}$ contracts the boundary of the n-ball D^n to a point on the n-sphere"

Since D^n compact and S^n Hausdorff $\Rightarrow \tilde{f}$ closed map, in-fact a closed quotient map.

Quotient Topology

Def^n/ Let X, Y be topological spaces and consider $f : X \to Y$ an onto map. Then we define the <u>Quotient Topology</u> on Y to be the only topology on Y that makes f an identification, and the finest topology for which f is continuous.

What follows next is what I mainly think of when I think of Quotient Top.

Def^n/ Suppose $\sim$ is an equivalence relation on X and let $\pi : X \to X/\sim$ be the natural map $\pi(x) = [x] = \{y \in X \mid x \sim y\}$. Then the topological space $X/\sim$ given the quotient topology in which π is a quotient map is called <u>QUOTIENT SPACE</u>

<u>Proposition</u>: Let $f : X \to Y$ be continuous map and $\sim$ an equivalence relation on X and $\pi : X \to X/\sim$ the canonical quotient map $\pi(x) = [x]$. There exists a continuous mapping $g : X/\sim \to Y$ such that $g \circ \pi = f$ iff f is constant on $\sim$ equivalence classes

$\boxed{\text{E5}}$ Let $f : X \to Y$ be continuous then we can study the fiber equivalence given by $x \sim y$ iff $f(x) = f(y)$. Notice $f^{-1}(c) = \{x \in X \mid f(x) = c\} = [x]$ where $[x] = \{y \in X \mid x \sim y\} = \{y \in X \mid f(x) = f(y)\}$. By the proposition above we induce from f a continuous 1-1 map on the quotient, $\bar{f} : X/\sim \to Y$. We note f is an identification iff $\bar{f}$ is homeomorphism.

Def^n/ Let $A \subseteq X$ and consider the "smallest" equivalence relation on X taking A as an equivalence class

$$x \sim y \iff x = y \text{ or } x, y \in A$$

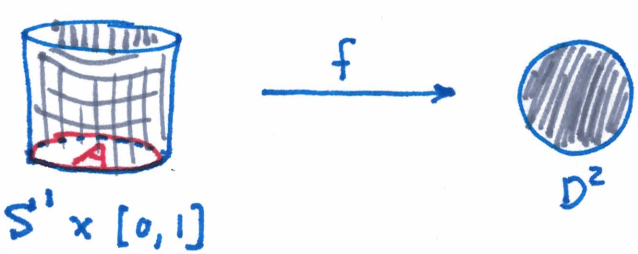
We denote $X/\sim$ by X/A.

$\boxed{E6}$ Consider $f : S^{n-1} \times [0,1] \longrightarrow D^n$ where $f(x,t) = tx$ observe this map is continuous and onto. Since $S^{n-1} \times [0,1]$ is <u>compact</u> and D^n is Hausdorff we have f is <u>closed map</u> and thus f is a closed identification. Notice f is <u>not</u> injective, but its fibers are solutions of $tx = y$ where $y \in D^n$ and $(x,t) \in S^{n-1} \times [0,1]$

Notice $x \in S^{n-1}$ gives $\|x\| = 1$ thus $\|tx\| = |t|\,\|x\| = \|y\| \Rightarrow |t| = \|y\|$ but $t \in [0,1]$ thus $|t| = t$ $\therefore$ $t = \|y\|$ and $\|y\| x = y$ $\therefore$ $x = \frac{y}{\|y\|}$ hence we calculate $f^{-1}\{y\} = \{(\frac{1}{\|y\|} y, \|y\|)\}$ provided $y \neq 0$. However, $f^{-1}\{0\} = S^{n-1} \times \{0\}$ (which is where injectivity is lost)

In summary, for $A = S^{n-1} \times \{0\}$ we have $z \sim w$ iff $z = w$ or $z, w \in A$ hence

$$\bar{f} : X/\sim \longrightarrow Y \Rightarrow \bar{f} : \frac{S^{n-1} \times [0,1]}{S^{n-1} \times \{0\}} \longrightarrow D^n \text{ homeomorphism}$$



- If we contract the base $S^1 \times [0,1]$ of the cylinder to a point then the contracted space is homeomorphic to disk D^2
- Don't ask me for the picture when $n \geq 3$.

$\boxed{E7}$ $X = [0,1]^n \subset \mathbb{R}^n$ has boundary $\partial X = [0,1]^n - (0,1)^n$. Let $X = I^n$, $I = [0,1]$

The quotient space $I^n/\partial I^n$ is interesting to study.

Recall $I^n \cong D^n$ and in $\boxed{E4}$ on 4 $f : D^n \to S^n$ was an identification which induced a homeomorphism

$$\tilde{f} : \underbrace{\{x \in D^n \mid \|x\| < 1\}}_{\text{this is } \approx\ D^n/\partial D^n} \to \{(x,y) \in S^n \mid y < 1\} = S^n - \{(0,1)\}$$

$(0,1) = (\underbrace{0,0,\ldots,0}_{n\text{-fold}},1)$

$f : X \to Y$ continuous and f an identification then

$\bar{f} : X/\sim \ \to Y$ is a homeomorphism

$$f(x) = \left(2x\sqrt{1-\|x\|^2},\ 2\|x\|^2 - 1\right)$$

$$x \sim y \iff \begin{pmatrix} 2x\sqrt{1-\|x\|^2} = 2y\sqrt{1-\|y\|^2} \\ 2\|x\|^2 - 1 = 2\|y\|^2 - 1 \end{pmatrix} \iff \|x\| = \|y\| \ \& \ x\sqrt{1-\|x\|^2} = y\sqrt{1-\|y\|^2}$$

$$\iff \|x\| = \|y\| \ \& \ x = y \text{ where } \|x\| \neq 1 \text{ or } \|x\| = \|y\| = 1$$

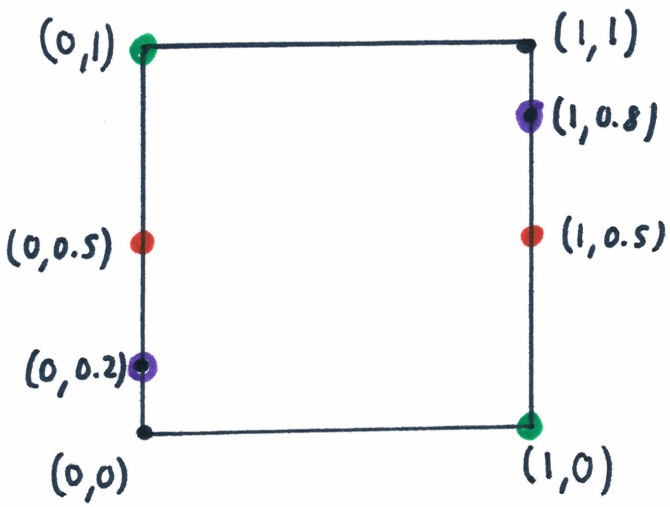
For $\|x\| = 1$,

$$f^{-1}\{x\} = \{x \in D^n \mid \|x\| = 1\} = \partial D^n$$

$$\bar{f} : D^n/\partial D^n \to S^n \text{ is homeomorphism}$$

Then $$I^n/\partial I^n \cong D^n/\partial D^n \cong S^n$$

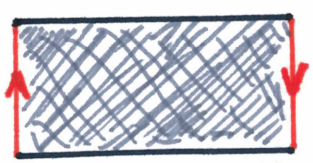
$\boxed{E8}$ Möbius Strip



$X = [0,1]^2$ and form the following equivalence, every pt $[P] = \{P\}$ except for

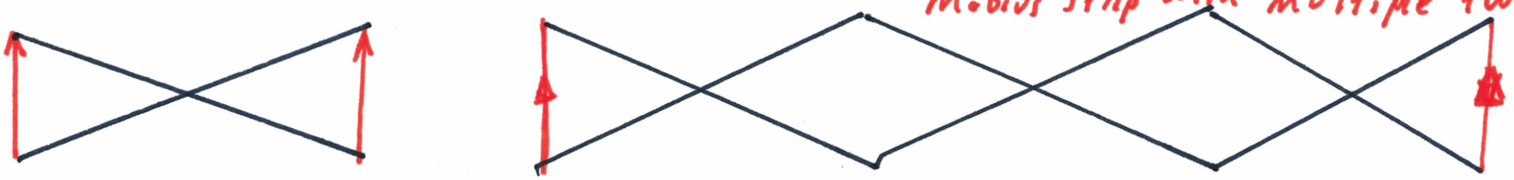
$$(0,y) \sim (1, 1-y) \quad \text{for each } y \in [0,1]$$

this means $X/\sim$ glues the ends top/bottom and bottom/top



← another method to visualize the Möbius strip

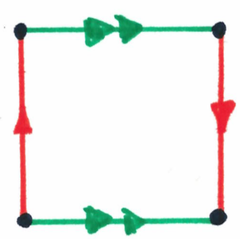
(I'm not sure what the right picture is for Mobius strip with multiple twists.)



$\boxed{E9}$ Klein Bottle

$$\frac{[0,1]\times[0,1]}{\sim}$$

where $\underbrace{(0,y) \sim (1, 1-y)}_{\text{twist and glue the vertical ends}}$ and $\underbrace{(x,0) \sim (x,1)}_{\text{glue the top/base}}$ for all $x,y \in [0,1]$



$(0,0) \sim (1,1)$
$(0,1) \sim (1,0)$

$(0,0) \sim (0,1)$
$(1,0) \sim (1,1)$

In total $(0,0) \sim (0,1) \sim (1,0) \sim (1,1)$

Remark: Quotient of compact space is compact, Quotient of connected space is likewise connected... Hausdorff... not so nice,

Th^{m}/ Let $f : X \longrightarrow Y$ be an identification, where X is compact and Hausdorff. TFAE,

(1.) Y is Hausdorff

(2.) f is closed identification

(3.) $K = \{ (x_1, x_2) \in X \times X \mid f(x_1) = f(x_2) \}$ is closed in the product.

Proof: see Manetti pg. 91.