LECTURE 19 : DERIVATIVES

4. DIFFERENTIATION

In this chapter, we discuss basic properties of the derivative of a function and several major theorems, including the Mean Value Theorem and l'Hôpital's Rule.

4.1 DEFINITION AND BASIC PROPERTIES OF THE DERIVATIVE

Let G be an open subset of $\mathbb{R}$ and consider a function $f\colon G \to \mathbb{R}$. For every $a \in G$, the function

$$\phi_a(x) = \frac{f(x)-f(a)}{x-a} = \frac{\Delta y}{\Delta x}$$

is defined on $G \setminus \{a\}$. Since G is an open set, a is a limit point of $G \setminus \{a\}$ (see Example 2.6.6). Thus, it is possible to discuss the limit

$$\lim_{x\to a} \phi_a(x) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

Definition 4.1.1 Let G be an open subset of $\mathbb{R}$ and let $a \in G$. We say that the function f defined on G is *differentiable at a* if the limit

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

exists (as a real number). In this case, the limit is called the *derivative of f at a* denoted by $f'(a)$, and f is said to be *differentiable at a*. Thus, if f is differentiable at a, then

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

We say that f is *differentiable on G* if f is differentiable at every point $a \in G$. In this case, the function $f'\colon G \to \mathbb{R}$ is called the derivative of f on G.

$x \mapsto f'(x)$ for $x \in G$.

Example 4.1.1 (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x$ and let $a \in \mathbb{R}$. Then

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x - a}{x - a} = \lim_{x \to a} 1 = 1.$$

It follows that f is differentiable at a and $f'(a) = 1$.

(b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$ and let $a \in \mathbb{R}$. Then

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} \frac{(x - a)(x + a)}{x - a} = \lim_{x \to a} (x + a) = 2a.$$

Thus, f is differentiable at every $a \in \mathbb{R}$ and $f'(a) = 2a$.

(c) Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = |x|$ and let $a = 0$. Then

$$f(x) = \begin{cases} -x & : x \le 0 \\ x & : x \ge 0 \end{cases}$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1,$$

$$f'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

and

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1.$$

Therefore, $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist and, hence, f is not differentiable at 0.

Theorem 4.1.1 Let G be an open subset of $\mathbb{R}$ and let f be defined on G. If f is differentiable at $a \in G$, then f is continuous at this point.

Proof: We have the following identity for $x \in G \setminus \{a\}$:

$$f(x) = f(x) - f(a) + f(a)$$
$$= \frac{f(x) - f(a)}{x - a}(x - a) + f(a).$$

$$\lim_{x \to a} f(x) = \overbrace{\lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right)}^{f'(a)} \lim_{x \to a} \cancelto{0}{(x - a)} + \lim_{x \to a} f(a) = f(a)$$

Thus,

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a}(x - a) + f(a) \right] = f'(a) \cdot 0 + f(a) = f(a).$$

Therefore, f is continuous at a by Theorem 3.3.2. □

Remark 4.1.2 The converse of Theorem 4.1.1 is not true. For instance, the absolute value function $f(x) = |x|$ is continuous at 0, but it is not differentiable at this point (as shown in the example above).

Theorem 4.1.3 Let G be an open subset of $\mathbb{R}$ and let $f, g\colon G \to \mathbb{R}$. Suppose both f and g are differentiable at $a \in G$. Then the following hold.

(a) The function $f + g$ is differentiable at a and

$$(f + g)'(a) = f'(a) + g'(a).$$

(b) For a constant c, the function cf is differentiable at a and

$$(cf)'(a) = cf'(a).$$

(c) The function fg is differentiable at a and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

(d) Suppose additionally that $g(a) \neq 0$. Then the function $\frac{f}{g}$ is differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

Proof: The proofs of (a) and (b) are straightforward and we leave them as exercises. Let us prove (c). For every $x \in G \setminus \{a\}$, we can write

$$\begin{aligned}\frac{(fg)(x) - (fg)(a)}{x-a} &= \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x-a} \\ &= \frac{(f(x) - f(a))g(x)}{x-a} + \frac{f(a)(g(x) - g(a))}{x-a}.\end{aligned}$$

$$\lim_{x \to a} (g(x)) = g(a)$$

By Theorem 4.1.1, the function g is continuous at a and, hence,

$$\lim_{x \to a} g(x) = g(a). \tag{4.1}$$

Thus,

$$\lim_{x \to a} \frac{(fg)(x) - (fg)(a)}{x-a} = f'(a)g(a) + f(a)g'(a).$$

This implies (c).

Next we show (d). Since $g(a) \neq 0$, by (4.1), there exists an open interval I containing a such that $g(x) \neq 0$ for all $x \in I$. Let $h = \frac{f}{g}$. Then h is defined on I. Moreover,

$$\begin{aligned}\frac{h(x) - h(a)}{x-a} &= \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(x)} + \frac{f(a)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} \\ &= \frac{\frac{1}{g(x)}(f(x) - f(a)) + \frac{f(a)}{g(x)g(a)}(g(a) - g(x))}{x-a} \\ &= \frac{1}{g(x)g(a)}\left[g(a)\frac{f(x) - f(a)}{x-a} - f(a)\frac{g(x) - g(a)}{x-a}\right].\end{aligned}$$

$$f = hg$$
$$f'(a) = h'(a)g(a) + h(a)g'(a)$$
$$h'(a) = \frac{f'(a) - h(a)g'(a)}{g(a)} = \frac{g(a)f'(a) - f(a)g'(a)}{(g(a))^2}$$

Taking the limit as $x \to a$, we obtain (d). The proof is now complete. □

Example 4.1.2 Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$ and let $a \in \mathbb{R}$. Using Example 4.1.1(a) and Theorem 4.1.3(c) we can provide an alternative derivation of a formula for $f'(a)$. Indeed, let $g\colon \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x$. Then $f = g \cdot g$ so

$$f'(a) = (gg)'(a) = g'(a)g(a) + g(a)g'(a) = 2g'(a)g(a) = 2a.$$

Proceeding by induction, we can obtain the derivative of $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^n$ for $n \in \mathbb{N}$ as $g'(a) = nx^{n-1}$. Furthermore, using this and Theorem 4.1.3(a)(b) we obtain the familiar formula for the derivative of a polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$ as $p'(x) = na_n x^{n-1} + \cdots + 2a_2 x + a_1$.

$$f(x) = f(a) + f'(a)(x-a) + \overbrace{u(x)(x-a)}^{\eta_f(x)} \qquad \eta_f(x) \longrightarrow 0 \text{ as } x \longrightarrow a.$$

$$g(x) = g(a) + g'(a)(x-a) + \underbrace{v(x)(x-a)}_{\eta_g(x) \longrightarrow 0 \text{ as } x \rightarrow a.}$$

$$f(x)g(x) = \big(f(a) + f'(a)(x-a) + \eta_f(x)\big)\big(g(a) + g'(a)(x-a) + \eta_g(x)\big)$$

$$= f(a)g(a) + \boxed{[f'(a)g(a) + f(a)g'(a)]}(x-a) + \underbrace{\eta_f(x)g(x) + f(x)\eta_g(x)}_{\underbrace{\text{function}(x)}_{\text{"}u(x)\text{" for } f(x)\cdot g(x).}(x-a)}$$

capturing departure of f(x) from tangent

The following lemma is very convenient for studying the differentiability of the composition of functions.

$f(x) = f(a) + f'(a)(x-a) + u(x)(x-a)$

line at $x=a$

Lemma 4.1.4 Let G be an open subset of $\mathbb{R}$ and let $f\colon G \to \mathbb{R}$. Suppose f is differentiable at a. Then there exists a function $u\colon G \to \mathbb{R}$ satisfying

$$f(x) - f(a) = [f'(a) + u(x)](x-a) \text{ for all } x \in G$$

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and $\lim_{x\to a} u(x) = 0$.

Proof: Define

$$u(x) = \begin{cases} \dfrac{f(x)-f(a)}{x-a} - f'(a), & x \in G \setminus \{a\} \\ 0, & x = a. \end{cases}$$

Since f is differentiable at a, we have

$$\lim_{x\to a} u(x) = \lim_{x\to a} \left(\frac{f(x)-f(a)}{x-a} - f'(a) \right) = f'(a) - f'(a) = 0.$$

Therefore, the function u satisfies the conditions of the lemma. □

Theorem 4.1.5 — Chain rule. Let $f\colon G_1 \to \mathbb{R}$ and let $g\colon G_2 \to \mathbb{R}$, where G_1 and G_2 are two open subsets of $\mathbb{R}$ with $f(G_1) \subset G_2$. Suppose f is differentiable at a and g is differentiable at $f(a)$. Then the function $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Proof: Since f is differentiable at a, by Lemma 4.1.4, there exists a function u defined on G_1 with

$$f(x) - f(a) = [f'(a) + u(x)](x-a) \text{ for all } x \in G_1,$$

and $\lim_{x\to a} u(x) = 0$.

Similarly, since g is differentiable at $f(a)$, there exists a function v defined on G_2 with

$$g(t) - g(f(a)) = [g'(f(a)) + v(t)][t - f(a)] \text{ for all } t \in G_2, \tag{4.2}$$

and $\lim_{t\to f(a)} v(t) = 0$.

Applying (4.2) for $t = f(x)$, we have

$$g(f(x)) - g(f(a)) = [g'(f(a)) + v(f(x))][f(x) - f(a)].$$

Composite of lines has a slope formed by product of slope.

Thus,

$$g(f(x)) - g(f(a)) = [g'(f(a)) + v(f(x))][f'(a) + u(x)](x-a) \text{ for all } x \in G_1.$$

$f(x) = m_1 x + b_1$

This implies

$g(x) = m_2 x + b_2$

$$\frac{g(f(x)) - g(f(a))}{x-a} = [g'(f(a)) + v(f(x))][f'(a) + u(x)] \text{ for all } x \in G_1 \setminus \{a\}.$$

$(f \circ g)(x) = f(m_2 x + b_2)$

By the continuity of f at a and the property of v, we have $\lim_{x\to a} v(f(x)) = 0$ and, hence,

$= m_1(m_2 x + b_2) + b_1$

$$\lim_{x\to a} \frac{g(f(x)) - g(f(a))}{x-a} = g'(f(a))f'(a).$$

$= m_1 m_2 x + m_1 b_2 + b_1$

The proof is now complete. □

■ **Example 4.1.3** Consider the function $h\colon \mathbb{R} \to \mathbb{R}$ given by $h(x) = (3x^4 + x + 7)^{15}$. Since $h(x)$ is a polynomial we could in principle compute $h'(x)$ by expanding the power and using Example 4.1.2. However, Theorem 4.1.5 provides a shorter way. Define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x^4 + x + 7$ and $g(x) = x^{15}$. Then $h = g \circ f$. Given $a \in \mathbb{R}$, it follows from Theorem 4.1.5 that

$$(g \circ f)'(a) = g'(f(a))f'(a) = 15(3a^4 + a + 7)^{14}(12a^3 + 1).$$

■ **Example 4.1.4** By iterating the Chain Rule, we can extended the result to the composition of more than two functions in a straightforward way. For example, given functions $f\colon G_1 \to \mathbb{R}$, $g\colon G_2 \to \mathbb{R}$, and $h\colon G_3 \to \mathbb{R}$ such that $f(G_1) \subset G_2$, $g(G_2) \subset G_3$, f is differentiable at a, g is differentiable at $f(a)$, and h is differentiable at $g(f(a))$, we obtain that $h \circ g \circ f$ is differentiable at a and $(h \circ g \circ f)'(a) = h'(g(f(a)))g'(f(a))f'(a)$.

Definition 4.1.2 Let G be an open set and let $f\colon G \to \mathbb{R}$ be a differentiable function. If the function $f'\colon G \to \mathbb{R}$ is also differentiable, we say that f is *twice differentiable* (on G). The second derivative of f is denoted by f'' or $f^{(2)}$. Thus, $f'' = (f')'$. Similarly, we say that f is three times differentiable if $f^{(2)}$ is differentiable, and $(f^{(2)})'$ is called the third derivative of f and is denoted by f''' or $f^{(3)}$. We can define in this way *n times differentiability* and the *nth* derivative of f for any positive integer n. As a convention, $f^{(0)} = f$.

Definition 4.1.3 Let I be an open interval in $\mathbb{R}$ and let $f\colon I \to \mathbb{R}$. The function f is said to be *continuously differentiable* if f is differentiable on I and f' is continuous on I. We denote by $C^1(I)$ the set of all continuously differentiable functions on I. If f is n times differentiable on I and the nth derivative is continuous, then f is called *n times continuously differentiable*. We denote by $C^n(I)$ the set of all n times continuously differentiable functions on I.

Exercises

4.1.1 Prove parts (a) and (b) of Theorem 4.1.3.

4.1.2 Compute the following derivatives directly from the definition. That is, do not use Theorem 4.1.3, but rather compute the appropriate limit directly (see Example 4.1.1).

(a) $f(x) = mx + b$ where $m, b \in \mathbb{R}$.

(b) $f(x) = \dfrac{1}{x}$ (here assume $x \neq 0$).

(c) $f(x) = \sqrt{x}$ (here assume $x > 0$)

4.1.3 Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^2, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases}$$

(a) Prove that f is differentiable at 0. Find $f'(x)$ for all $x \in \mathbb{R}$.

(b) Is f' continuous? Is f' differentiable?

4.1.4 Let

$$f(x) = \begin{cases} x^\alpha, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases}$$

(a) Determine the values of α for which f is continuous on $\mathbb{R}$.

(b) Determine the values of α for which f is differentiable on $\mathbb{R}$. In this case, find f'.

4.1.5 Use Theorems 4.1.3 and 4.1.5 to compute the derivatives of the following functions at the indicated points (see also Example 4.1.4). (Assume known that the function $\sin x$ is differentiable at all points and that its derivative is $\cos x$.)

(a) $f(x) = \dfrac{3x^4+7x}{2x^2+3}$ at $a=-1$.

(b) $f(x) = \sin^5(3x^2+\frac{\pi}{2}x)$ at $a=\frac{\pi}{8}$

4.1.6 Determine the values of x at which each function is differentiable.

(a) $f(x) = \begin{cases} x\sin\dfrac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$

(b) $f(x) = \begin{cases} x^2\sin\dfrac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$

4.1.7 Determine if each of the following functions is differentiable at 0. Justify your answer.

(a) $f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q}; \\ x^3, & \text{if } x \notin \mathbb{Q}. \end{cases}$

(b) $f(x) = [x]\sin^2(\pi x)$.

(c) $f(x) = \cos(\sqrt{|x|})$.

(d) $f(x) = x|x|$.

4.1.8 Let f, g be differentiable at a. Find the following limits:

(a) $\lim_{x\to a} \dfrac{xf(a)-af(x)}{x-a}$.

(b) $\lim_{x\to a} \dfrac{f(x)g(a)-f(a)g(x)}{x-a}$.

4.1.9 Let G be an open subset of $\mathbb{R}$ and $a \in G$. Prove that if $f\colon G \to \mathbb{R}$ is Lipschitz continuous, then $g(x) = (f(x)-f(a))^2$ is differentiable at a.

4.1.10 ▷ Let f be differentiable at a and $f(a) > 0$. Find the following limit:

$$\lim_{n\to\infty} \left(\frac{f(a+\frac{1}{n})}{f(a)} \right)^n.$$

4.1.11 ▶ Consider the function

$$f(x) = \begin{cases} x^2\sin\dfrac{1}{x} + cx, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0, \end{cases}$$