LECTURE 19 : PROJECTIVE SPACES

We follow §5.3 and §5.4 which present projective spaces as they arise from a quotient space corresponding to a partition of the space by homeomorphisms of a particular type. We begin with,

Def/ Homeo(X) is the set of homeomorphisms from a topological space X to itself.

Since the composite of homeomorphisms and the inverse of a homeomorphism is once again a homeomorphism and $Id_X : X \to X$ where $Id_X(x) = x$ $\forall x \in X$ has $Id_X \in$ Homeo(X) and in fact:

Thm/ Homeo(X) forms a group under composition of functions

The group Homeo(X) is usually huge, but subgroups $G \leq$ Homeo(X) are of particular use,

Defn/ Let $G \leq$ Homeo(X) and <u>define</u> $x \sim y$ if $\exists g \in G$ s.t $y = g(x)$, for any $x, y \in X$. This is an equivalence relation on X whose equivalence classes are known as G-orbits and X/G is the corresponding quotient space.

Proposition: Let $G \subseteq \mathrm{Homeo}(X)$ be a subgroup of homeomorphisms on X and let $\pi : X \to X/G$ be the canonical quotient map. Then π is an open map. If G is a finite group then π is also closed map.

Proof: Consider subset $A \subseteq X$, the inverse image under π is union of G-orbits,

$$\pi^{-1}(\pi(A)) = \bigcup \{ g(A) \mid g \in G \}$$

If A is open then $g(A)$ is open $\Rightarrow$ $\pi^{-1}(\pi(A))$ is union of open sets hence $\pi^{-1}(\pi(A))$ is open. Thus, by def$^{\underline{n}}$ of quotient topology, $\pi(A)$ is open. If G is finite then argument above holds for closed set A since union of finitely many closed sets is again closed.//

Proposition: Let G be a group of homeomorphisms of X. The quotient X/G is Hausdorff $\iff K = \{(x, g(x)) \mid x \in X, g \in G\}$ is closed in $X \times X$.

Proof: The map $\pi : X \to X/G$ is an open surjection. Thus define $P(x,y) = (\pi(x), \pi(y))$ as an open surjection of $X \times X \xrightarrow[P]{} X/G \times X/G$. So, P is an <u>identification</u>. Observe $P(x,y) \in \Delta(X/G)$ iff $x \sim y$ and $x \sim y$ iff $(x,y) \in K$. Thus $P^{-1}(\Delta(X/G)) = K$. Finally, as P is an identification $\Delta(X/G)$ is closed iff K is closed. But, recall $\Delta(X/G)$ is closed iff X/G is Hausdorff.//

(Manetti, pg. 93, following pretty much verbatim)

PROPOSITION: Let G be subgroup of $Homeo(X)$ where X is Hausdorff and $\pi : X \to X/G$ is canonical map. Suppose $\exists$ open $A \subseteq X$ such that

(1.) the quotient map $\pi : A \to X/G$ is onto

(2.) $\{g \in G \,/\, g(A) \cap A \neq \emptyset\}$ is finite

Then X/G is Hausdorff

<u>Proof</u>: suppose $\{g \in G \,/\, g(A) \cap A \neq \emptyset\}$ is finite then $\exists$ finitely many $g_1, g_2, \dots, g_n \in G$ for which $g_i(A) \cap A \neq \emptyset$ where $1 \leq i \leq n$. Let $p, q \in X/G$ with $p \neq q$. Further choose $x, y \in A$ for which $\pi(x) = p$ and $\pi(y) = q$ (x & y are in different G-orbits) Since X is assumed Hausdorff, for any $i = 1, 2, \dots, n$ we may select open U_i, V_i subsets of X where $x \in U_i$, $g_i(y) \in V_i$ and $U_i \cap V_i = \emptyset$. Then construct

$$U = A \cap \left(\bigcap_{i=1}^{n} U_i \right) \quad \text{and} \quad V = A \cap \left(\bigcap_{i=1}^{n} g_i^{-1}(V_i) \right)$$

from which <u>Manetti argues</u> $U \cap g(V) = \emptyset$ for all $g \in G$, I believe we can see $p \in \pi(U)$ and $q \in \pi(V)$ with $\pi(U) \cap \pi(V) = \emptyset$ hence X/G Hausdorff. //

(p. 94)

$\boxed{E1}$ Let $G \subseteq \text{Homeo}(\mathbb{R}^n)$ be the group of translations by vectors with integer components; $G = \{ \varphi : \mathbb{R}^n \to \mathbb{R}^n \mid \varphi(x) = x + z,\ z \in \mathbb{Z}^n \}$.
Form the quotient space $\mathbb{R}^n/G$. We can show $\mathbb{R}^n/G$ is Hausdorff.
Consider,

$$A = \{ (x_1, \dots, x_n) \mid |x_i| < 1 \ \forall i = 1, 2, \dots, n \} \subset \mathbb{R}^n \qquad A = (-1,1)^n$$

and notice $\pi : A \to \mathbb{R}^n/G$ is surjective

($\mathbb{R}^n/G$ is a lattice where A is a fundamental region, take any point $(x_1, x_2, \dots, x_n) \in A$ then $G(x_1, \dots, x_n) = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{\mathbb{Z}^n} + (x_1, \dots, x_n)$)

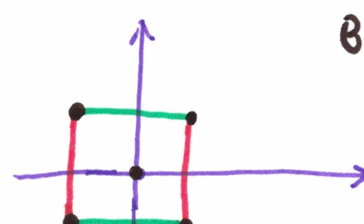
not quite, too big.

If $z \in \mathbb{Z}^n$, then $A \cap (A + z) \neq \emptyset$ only if $|a_i| \leq 1$ for $i = 1, 2, \dots, n$.
Thus by Prop. on pg. 3 (aka Prop 5.17 on p. 93 of Manetti) we find $\mathbb{R}^n/G$ is Hausdorff.

Corollary: Let G be a finite group of homeomorphisms of a Hausdorff space. Then X/G is Hausdorff.

Proof: apply Prop. on p. 3 with $A = X$. //

$\boxed{E2}$ $\mathbb{R}^2/\mathbb{Z}^2$ = TORUS, well... ?



But...
$(-\tfrac{1}{2}, -\tfrac{1}{2}) + (1,1) = (\tfrac{1}{2}, \tfrac{1}{2})$
need smaller choice of A, nevermind *2

* Remark: $g(A) \cap A \neq \emptyset$ for finitely many $g \in G$. Ok, so, I guess it's ok just $g = (1,0), (-1,0), (1,1), (-1,1), (1,-1), (-1,-1)$.

PROJECTIVE SPACES

Def^n/ Homotheties or Dilations on $\mathbb{R}^{n+1}$ are mappings $\varphi(x) = \lambda x$ for some $\lambda \in \mathbb{R} - \{0\}$. Projective space $P^n(\mathbb{R}) = \dfrac{\mathbb{R}^{n+1} - \{0\}}{G}$ where G is the group of dilations on $\mathbb{R}^{n+1}$. Here $x \sim y \iff x = \lambda y$ for some $\lambda \in \mathbb{R} - \{0\}$. $P^n(\mathbb{R})$ is given quotient topology.

The G-orbits in $\mathbb{R}^{n+1} - \{0\}$ are lines through the origin, with the origin deleted.

$$[x_0, x_1, \dots, x_n] = \{(y_0, \dots, y_n) \in \mathbb{R}^{n+1} - \{0\} \mid (x_0, \dots, x_n) = \lambda(y_0, \dots, y_n)\}$$

$\pi : \mathbb{R}^{n+1} - \{0\} \longrightarrow P^n(\mathbb{R})$ where $\pi(x) = [x]$ is an open identification

The inclusion $i : S^n \longrightarrow \mathbb{R}^{n+1} - \{0\}$ can be composed with π

$$\begin{tikzcd} S^n \arrow[r, "i"] \arrow[d, "\pi'"] & \mathbb{R}^{n+1} - \{0\} \arrow[r, "r"] \arrow[d, "\pi"] & S^n \arrow[d, "\pi'"] \\ S^n/\sim \arrow[r, "f"] & P^n(\mathbb{R}) \arrow[r, "f^{-1}"] & S^n/\sim \end{tikzcd}$$

$r(x) = \dfrac{x}{\|x\|}$, the normalizing map $r(x) = \hat{x}$

- It appears Manetti is defining f and f^{-1} with the above diagram, well maybe not, he assumes f is a continuous bijection, Nevermind this all goes back to the 'universal prop of identification' it gives the existence of such a map.

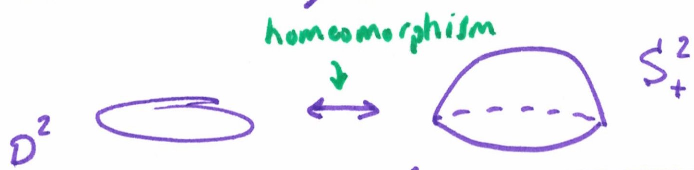
Def^n/ $x, y \in S^n$ have $x \sim y$ if $y = \pm x$ then $\pi' : S^n \longrightarrow S^n/\sim$ has $\pi'(x) = \{-x, x\}$

$$\begin{tikzcd} S^n \arrow[r, "i"] \arrow[d, "\pi'"] & \mathbb{R}^{n+1} - \{0\} \arrow[r, "\pi"] & P^n(\mathbb{R}) \\ S^n/\sim \arrow[urr, dashed, "f"] & & \end{tikzcd}$$

Thm: $S^n/\sim \;\cong\; P^n(\mathbb{R})$ & $P^n(\mathbb{R})$ is Hausdorff

<u>Proof</u>: Let $\varphi(\{x,-x\}) = [x]$ and note $[x]=[-x]$ as $x\sim -x$ $(\lambda=-1)$ thus φ is well-defined. If $[x]\in P^n(\mathbb{R})$ then $x\neq 0$ and thus $r(x)=\hat{x}\in S^n$ and we note $x=\|x\|\,r(x)$ and $\varphi(\{r(x),-r(x)\}) = [r(x)] = [x]$ $\therefore$ φ onto.
Suppose $\varphi(\{x,-x\}) = \varphi(\{y,-y\})$ then $[x]=[y] \Rightarrow y=\lambda x$. However,
$1=\|x\|=\|y\|$ & $\|y\| = |\lambda|\,\|x\| \Rightarrow |\lambda|=1 \;\therefore\; \lambda=\pm 1 \;\therefore\; y=\pm x \Rightarrow \varphi$ 1-1.
Thus φ is a bijection. Then think of $\varphi = f$ for diagrams on 5 and applying Prop. 5.8 gives continuity of f and also f^{-1}.
Note $S^n/\sim \;=\; \dfrac{S^n}{\{\pm Id\}}$ thus as S^n is Hausdorff & $|\{\pm Id\}|=2$ we find $S^n/\sim$ is Hausdorff $\Rightarrow P^n(\mathbb{R})$ is Hausdorff. //

$\boxed{E3}$ Consider $D^2 = \{x\in\mathbb{R}^2 \mid \|x\|\le 1\}$. Let $x\sim y$ iff $x=y$ or $\|x\|=\|y\|=1,\ x=-y$ then set $X = D^2/\sim$. We can show $X \cong P^2(\mathbb{R})$



$(x,y) \longleftrightarrow (x,y,\sqrt{1-x^2-y^2})$

$(x,y,0)\sim(-x,-y,0)$ gives $S^2_+/\sim$

Then include $S^2_+\subseteq S^2$ by $i: S^2_+\to S^2$, let $\pi: S^2\to P^2(\mathbb{R})$, $p: S^2_+\to X$ be the canonical quotients.

E3 continued

$$\begin{tikzcd} S^2_+ \arrow[r, "i"] \arrow[d, "p"] & S^2 \arrow[d, "\pi"] \\ X \arrow[r, "j"] & P^2(\mathbb{R}) \end{tikzcd}$$

Commutative diagram of continuous maps

$$\pi \circ i = j \circ p$$

Here j is bijective and maps compact space X to Hausdorff $P^2(\mathbb{R})$ thus j is Homeomorphism.

$\boxed{E4}$ For $i = 0, 1, .., n$ define $A_i = \{ [x_0, .., x_n] \in P^n(\mathbb{R}) \mid x_i \neq 0 \}$ and note $A_i \cong \mathbb{R}^n$ is open in $P^n(\mathbb{R})$. Note $\{A_0, A_1, .., A_n\}$ serves as an identification cover. In particular, using $\pi : \mathbb{R}^{n+1} - \{0\} \to P^n(\mathbb{R})$ we have $\pi^{-1}(A_i) = \{ (x_0, ..., x_n) \in \mathbb{R}^{n+1} - \{0\} \mid x_i \neq 0 \}$ and the quotient map is open. Let $f : \mathbb{R}^n \longrightarrow A_0$ given by

$$f(y_1, .., y_n) = [1, y_1, ..., y_n]$$

is continuous and bijective. Observe

$$f^{-1} \circ \pi : \pi^{-1}(A_0) \longrightarrow \mathbb{R}^n \quad \text{has } (f^{-1} \circ \pi)(x_0, x_1, .., x_n) = \left(\frac{x_1}{x_0}, .., \frac{x_n}{x_0}\right)$$

$\Rightarrow f^{-1}$ continuous by universiality of identifications.

Def^{n}/ $\mathbb{C}^{n+1} - \{0\}$ has equivalence relation $x \sim y$ iff $\exists \lambda \in \mathbb{C} - \{0\}$ s.t. $x = \lambda y$

then $P^n(\mathbb{C}) = \dfrac{\mathbb{C}^n - \{0\}}{G}$ where $G = \{\varphi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}^{n+1} - \{0\} \,/\, \varphi(x) = \lambda x$ for some $\lambda \in \mathbb{C}, \lambda \neq 0$ $\forall x \in \mathbb{C}^{n+1} - \{0\}\}$

and $P^n(\mathbb{C})$ is given quotient toplogy

$\boxed{\text{Proposition: Projective spaces, both complex and real, are connected, compact and Hausdorff}}$

Proof: The projections $S^n \longrightarrow P^n(\mathbb{R})$ and $S^{2n+1} = \{z \in \mathbb{C}^{n+1} \,/\, \|z\| = 1\} \to P^n(\mathbb{C})$ are continuous and onto $\Rightarrow P^n(\mathbb{R}), P^n(\mathbb{C})$ are compact and connected since S^n is compact and connected. To see $P^n(\mathbb{C})$ is Hausdorff, study

$$K = \{(x,y) \in (\mathbb{C}^{n+1} - \{0\}) \times (\mathbb{C}^{n+1} - \{0\}) \mid x = \lambda y\}$$

this set is closed $\therefore$ $P^n(\mathbb{C})$ is Hausdorff (and we already argued $P^n(\mathbb{R})$ Hausdorff)